Propagation of acoustic and electromagnetic waves in piezoelectric, piezomagnetic, and magnetoelectric materials with tetragonal and hexagonal symmetry

G. Iadonisi, C. A. Perroni, G. Cantele, and D. Ninno
CNR–INFM Coherentia and Dipartimento di Scienze Fisiche, Università di Napoli "Federico II," Complesso Universitario Monte S. Angelo, Via Cinthia, 80126 Napoli, Italy
(Received 27 April 2009; revised manuscript received 2 July 2009; published 10 September 2009)

The propagation of the acoustic and the electromagnetic signals is studied in materials with tetragonal and hexagonal symmetries that are piezoelectric, piezomagnetic, or magnetoelectric. Three magnetic spatial symmetries are considered: 622 for hexagonal crystals and 422 and $4'22'$ for tetragonal crystals. The equations for the five modes (three mainly acoustic and two mainly electromagnetic) are solved both in the general case and in specific cases in which the material is only piezoelectric, piezomagnetic, or magnetoelectric. It is found that the piezomagnetic and the magnetoelectric coefficients and the magnetic permeability renormalize the elastic constants, the piezoelectric coefficients, and the dielectric tensor. The acoustic frequencies depend on the angle θ that the component of the wave vector in the *a*-*b* plane forms with the *a* axis. The main contribution to the electromagnetic modes derives only from the dielectric tensor and the magnetoelectric coefficients and it is independent of θ and of the piezoelectric and the piezomagnetic coefficients. Starting from the general equations, a method has been devised to study separately the acoustic and the electromagnetic solutions. It is found that a number of relations must be verified in order to have stability of the electromagnetic and the acoustic modes.

DOI: 10.1103/PhysRevB.80.094103 PACS number(s): 77.65.−j, 77.84.−s, 72.55.+s

I. INTRODUCTION

The piezoelectric effects are known since long time and today there are many review books on the topic,[1–5] whereas the piezomagnetic effects were discovered only in the last decades.[6] For a long time this problem attracted mainly the physicists devoted to the classification of the point and the space groups of the magnetic materials.[7] Only recently the discovery of materials with piezomagnetic properties and their connections with the magnetoelectricity and the piezoelectricity begins to clarify the roles that they can play in new devices, transducers, and sensors for acoustic, electric, and magnetic signals.[8–14] Also, the finding of multifunctional materials that show both piezoelectric and piezomagnetic effects is a goal of the recent research in materials science. It was found that crystals such as Pb_2MnO_4 (Ref. 15) could show simultaneously piezoelectric and piezomagnetic effects: the point and the spatial symmetry groups of the material were identified in the tetragonal point group and in the $P\bar{4}2_1c$ spatial group that becomes the $P\bar{4}2_1c'$ magnetic group,[7] for which the effects of the symmetry on the elastic, the piezoelectric, and the piezomagnetic coefficients are well known. Recently, the problem of the coexistence of the ferroelectricity and the ferromagnetism has been studied in multiferroic materials, considering both the elementary mechanisms and their effects.[16]

The propagation of the coupled acoustic and electromagnetic signals has been studied both in piezoelectric[17–23] and in multiferroic materials[24,25] with different symmetries. In this work we develop the theory of the propagation of the acoustic and the electromagnetic signals in materials with tetragonal and hexagonal point symmetries assuming that they are at the same time piezoelectric, piezomagnetic, and magnetoelectric. The magnetic spatial groups $4'22'$ and 422 have as point group the tetragonal one, whereas the magnetic spatial group 622 has as point group the hexagonal one. The starting idea to treat all these problems is the generalization of the piezoelectromagnetism theory developed in a previous work for the hexagonal ceramics,[26] based on Ref. 27. We find that there are five modes (three mainly acoustic and two mainly electromagnetic) that can propagate in the crystal. The three acoustic modes can be strongly coupled with the piezoelectric and the piezomagnetic coefficients, whereas the two electromagnetic modes are characterized mainly by the dielectric and the magnetic susceptibilities and the magnetoelectric coefficients. In the infinite medium, the wave vector has components (q_x,q_y,k) with the z axis directed along the c axis and the contribution of the piezoelectric and the piezomagnetic coefficients depends on the angle θ, such that $\tan\theta=\frac{q_y}{q_x}$. In any case, when the electric and the magnetic permeabilities are large, the electromagnetic regime is reached at lower frequencies. The piezomagnetic coefficients and the magnetic permeability modify the elastic coefficients of the material; the piezomagnetic coefficients, the magnetoelectric ones and the magnetic permeability renormalize the piezoelectric coefficients; finally, the magnetoelectric coefficients and the magnetic permeability modify the dielectric constants. A general method is introduced in order to separately calculate, with a high accuracy, the acoustic and the electromagnetic frequencies. It is found that, for all the considered symmetries, the square of the electromagnetic frequencies is positive if the renormalized dielectric constants are positive, but it can occur that the group velocity of the modes can be larger than the light velocity c, indicating that the mode is not stable. For all the symmetries considered in this work, the acoustical modes are stable except $P\bar{4}2_1c'$ that can show instability due to the piezomagnetic effects.

In Sec. II, starting from the constitutive equations, the general equations are found by separately considering each symmetry. In Sec. III the theory is elaborated by writing the set of independent equations and discussing the method used to separately calculate the acoustic and the electromagnetic frequencies. In Sec. IV we calculate and discuss the features of the solutions for the hexagonal and the tetragonal symmetries considering piezoelectric, piezomagnetic, and magnetoelectric materials and the general case. Finally, in Sec. V some remarks and conclusions are addressed.

II. EQUATIONS FOR THE PROPAGATION OF THE ACOUSTIC AND THE ELECTROMAGNETIC SIGNALS IN PIEZOELECTRIC, PIEZOMAGNETIC, AND MAGNETOELECTRIC MATERIALS

The general form of the functional F that allows the derivation of the constitutive equations for a material that is piezoelectric, piezomagnetic, and magnetoelectric is given by[5]

$$F=\frac{1}{2}\left[\sum_{ijkl} c_{ijkl}s_{ij}s_{kl}-\sum_{ij}\varepsilon_{ij}E_iE_j-\sum_{ij}\mu_{ij}H_iH_j\right]-\sum_{ijk}e_{ijk}E_is_{jk}-\sum_{ijk}z_{ijk}H_is_{jk}-\sum_{ij}\lambda_{ij}E_iH_j, \tag{1}$$

where the indices i, j, k, and l assume the values of 1, 2, and 3; c_{ijkl} are the elastic constants; E_i and H_i are the components of the electric and the magnetic fields; e_{ijk} and z_{ijk} are the piezoelectric and the piezomagnetic coefficients; ε_{ij}, μ_{ij}, and λ_{ij} are the components of the dielectric constant, the magnetic permeability, and the magnetoelectric tensors; and, finally, the quantities s_{ij} are defined as $s_{11}=\partial_x u$, $s_{22}=\partial_y v$, $s_{33}=\partial_z w$, $s_{23}=\frac{1}{2}(\partial_z v+\partial_y w)$, $s_{13}=\frac{1}{2}(\partial_z u+\partial_x w)$, and $s_{12}=\frac{1}{2}(\partial_y u+\partial_x v)$, where u, v, and w are the matter displacements along the x, the y, and the z directions and ∂_x, for instance, is the partial derivative with respect to x. The constitutive equations are found starting from F through

$$T_{ij}=\partial_{s_{ij}}F,$$

$$D_i=-\partial_{E_i}F,$$

$$B_i=-\partial_{H_i}F, \tag{2}$$

where T_{ij}, D_i, and B_i are the stress tensor, the electric displacement, and the magnetic induction fields. Using the Voigt compact notation and considering that the material has tetragonal or hexagonal symmetry, Eqs. (2) reduce to

$$T_1=T_{11}=c_{11}\partial_x u+c_{12}\partial_y v+c_{13}\partial_z w,$$

$$T_2=T_{22}=c_{12}\partial_x u+c_{11}\partial_y v+c_{13}\partial_z w,$$

$$T_3=T_{33}=c_{13}\partial_x u+c_{13}\partial_y v+c_{33}\partial_z w,$$

$$T_4=T_{23}=c_{44}(\partial_z v+\partial_y w)-e_{14}E_x-z_{14}H_x=T_{32},$$

$$T_5=T_{13}=c_{44}(\partial_z u+\partial_x w)\mp e_{14}E_y\mp z_{14}H_y=T_{31},$$

$$T_6=T_{12}=c_{66}(\partial_y u+\partial_x v)-e_{36}E_z-z_{36}H_z=T_{21}, \tag{3}$$

$$D_x=\varepsilon_{11}E_x+e_{14}(\partial_z v+\partial_y w)+\lambda_{11}H_x,$$

$$D_y=\varepsilon_{11}E_y\pm e_{14}(\partial_z u+\partial_x w)+\lambda_{11}H_y,$$

$$D_z=\varepsilon_{33}E_z+e_{36}(\partial_y u+\partial_x v)+\lambda_{33}H_z, \tag{4}$$

$$B_x=\mu_{11}H_x+z_{14}(\partial_z v+\partial_y w)+\lambda_{11}E_x,$$

$$B_y=\mu_{11}H_y\pm z_{14}(\partial_z u+\partial_x w)+\lambda_{11}E_y,$$

$$B_z=\mu_{33}H_z+z_{36}(\partial_y u+\partial_x v)+\lambda_{33}E_z. \tag{5}$$

The nonzero elastic coefficients for the tetragonal materials are c_{11}, c_{12}, c_{13}, $c_{21}=c_{12}$, $c_{22}=c_{11}$, $c_{23}=c_{13}$, $c_{31}=c_{13}$, $c_{32}=c_{13}$, c_{33}, c_{44}, $c_{55}=c_{44}$, and c_{66}. The same holds for the hexagonal symmetry, but with the additional condition $2c_{66}=c_{11}-c_{12}$. For materials with spatial group $P\bar{4}2_1c'$ the nonzero piezoelectric coefficients are e_{14}, $e_{25}=e_{14}$, and e_{36}; the nonzero piezomagnetic ones are z_{14}, $z_{25}=z_{14}$, and z_{36}; and for the other symmetries (tetragonal and hexagonal) it occurs that $e_{36}=z_{36}=0$, $e_{25}=-e_{14}$, and $z_{25}=-z_{14}$. Finally, in all cases the magnetoelectric coefficients λ_{ij} are all zero, except λ_{33} and $\lambda_{22}=\lambda_{11}$; the same occurs for the dielectric tensor ($\varepsilon_{ij}=0$, except ε_{33} and $\varepsilon_{22}=\varepsilon_{11}$) and the magnetic susceptibility ($\mu_{ij}=0$, except μ_{33} and $\mu_{22}=\mu_{11}$).[7] From Eqs. (5) it is easy to calculate $\mathbf{H}$ and then the magnetization $\mathbf{M}=\mathbf{B}-\mu_0\mathbf{H}$,

$$M_x=\left(1-\frac{1}{\bar{\mu}_{11}}\right)B_x+\frac{z_{14}}{\bar{\mu}_{11}}(\partial_z v+\partial_y w)+\frac{\lambda_{11}}{\bar{\mu}_{11}}E_x,$$

$$M_y=\left(1-\frac{1}{\bar{\mu}_{11}}\right)B_y\pm\frac{z_{14}}{\bar{\mu}_{11}}(\partial_z u+\partial_x w)+\frac{\lambda_{11}}{\bar{\mu}_{11}}E_y,$$

$$M_z=\left(1-\frac{1}{\bar{\mu}_{33}}\right)B_z+\frac{z_{36}}{\bar{\mu}_{33}}(\partial_y u+\partial_x v)+\frac{\lambda_{33}}{\bar{\mu}_{33}}E_z, \tag{6}$$

where $\bar{\mu}_{ij}=\frac{\mu_{ij}}{\mu_0}$ and μ_0 is the vacuum magnetic permeability. The above constitutive equations are adapted to the groups of interest in the following way: (a) for the tetragonal spatial group 422 the bottom sign is taken in Eqs. (3)–(6) and furthermore $z_{36}=e_{36}=0$; (b) for the hexagonal spatial group the same rule as in (a) holds, with the further condition $2c_{66}=c_{11}-c_{12}$; and (c) for the tetragonal spatial group $4'22'$ the upper sign in the mentioned equations is taken. In Ref. 26 it was shown that the five acoustic and electromagnetic modes in a piezoelectric crystal can be found from equations in which the electromagnetic field is described through the vector and the scalar potentials $\mathbf{A}$ and Φ. For the systems of interest in this work those equations become

$$\partial_x T_{11}+\partial_y T_{21}+\partial_z T_{31}=\varrho\partial_{tt}u,$$

$$\partial_x T_{12}+\partial_y T_{22}+\partial_z T_{32}=\varrho\partial_{tt}v,$$

$$\partial_x T_{13}+\partial_y T_{23}+\partial_z T_{33}=\varrho\partial_{tt}w,$$

$$\nabla\cdot\mathbf{D}=0,$$

$$\nabla^2\mathbf{A} - \frac{1}{c^2}\partial_t^2\mathbf{A} = -\mu_0\partial_t\mathbf{P} - \nabla\times M,$$

$$\nabla^2\Phi - \frac{1}{c^2}\partial_t^2\Phi = \frac{1}{\varepsilon_0}\nabla\cdot\mathbf{P},$$

$$\mathbf{P} = \mathbf{D} - \varepsilon_0\mathbf{E},$$

$$\mathbf{M} = \mathbf{B} - \mu_0\mathbf{H},$$

$$\mathbf{B} = \nabla\times\mathbf{A},$$

$$\mathbf{E} = -\partial_t\mathbf{A} - \nabla\Phi,$$

$$\nabla\cdot\mathbf{A} + \frac{1}{c^2}\partial_t\Phi = 0, \tag{7}$$

where $\mathbf{P}$ is the polarization vector, ε_0 is the electric permeability, and $c=1/\sqrt{\varepsilon_0\mu_0}$ is the light velocity in the vacuum. The first three of Eqs. (7) are the dynamic equations for the matter displacements when mechanical and electromagnetic forces act; the fourth equation is the Gauss law in the material; the fifth and the sixth equations give the propagation of the vector and the scalar potentials, whose sources are the polarization and the magnetization currents and the polarization charge density; the seventh, the eighth, the ninth, and the tenth equations are the constitutive equations for the polarization and magnetization and the relations that connect the electric and the magnetic fields to the vector and the scalar potentials. Finally the last equation is the Lorentz condition. There is a number of equations (nine) larger than the unknown quantities (seven), indicating that two equations are depending on the others.

The starting point is writing the matter displacements $u(x,y,z)$, $v(x,y,z)$, and $w(x,y,z)$; the scalar and the vector potentials $\Phi(x,y,z)$ and $\mathbf{A}(x,y,z)$; and consequently the electric and the magnetic fields as

$$u(x,y,z) = [\partial_x\varphi(x,y) + \partial_y\psi(x,y)]\exp[i(kz-\omega t)],$$

$$v(x,y,z) = [\partial_y\varphi(x,y) - \partial_x\psi(x,y)]\exp[i(kz-\omega t)],$$

$$w(x,y,z) = W(x,y)\exp[i(kz-\omega t)],$$

$$\Phi(x,y,z) = \phi(x,y)\exp[i(kz-\omega t)],$$

$$\mathbf{A}(x,y,z) = \mathbf{F}(x,y)\exp[i(kz-\omega t)],$$

$$E_x(x,y,z) = (i\omega F_x - \partial_x\phi)\exp[i(kz-\omega t)],$$

$$E_y(x,y,z) = (i\omega F_y - \partial_y\phi)\exp[i(kz-\omega t)],$$

$$E_z(x,y,z) = i(\omega F_z - k\phi)\exp[i(kz-\omega t)] = -iM\exp[i(kz-\omega t)],$$

$$B_x(x,y,z) = (\partial_y F_z - ikF_y)\exp[i(kz-\omega t)],$$

$$B_y(x,y,z) = (-\partial_x F_z + ikF_x)\exp[i(kz-\omega t)],$$

$$B_z(x,y,z) = (\partial_x F_y - \partial_y F_x)\exp[i(kz-\omega t)] = -(\partial_{xx}+\partial_{yy})C_1\exp[i(kz-\omega t)]. \tag{8}$$

It turns out that the matter displacements are described through the functions $\varphi(x,y)$, $\psi(x,y)$, and $W(x,y)$ and the electromagnetic field is described through [we omit the factor $\exp i(kz-\omega t)$]

$$M = k\phi - \omega F_z, \quad G = \frac{\omega}{c^2}\phi - kF_z, \tag{9}$$

and the two-dimensional vector

$$\mathbf{F}_b \equiv (F_x,F_y) = \nabla\times[C_1(x,y)\hat{z}] + \nabla H(x,y) \tag{10}$$

with the $\hat{z}$ vector along the c axis. In Appendix A it is shown how to generalize the procedure used in the previous work[26] to find the independent equations that allow us to calculate the five modes (three acoustic and two electromagnetic) propagating in this material. This is done only for case (c) (tetragonal spatial group $4'22'$) that requires a more complicated mathematical procedure (the other cases can be treated similarly). For the symmetry 422 the following independent equations are obtained:

$$\nabla_b^2 A + 2\Delta c_{66}\partial_{xy}\psi_1 + ik(c_{13}+f_{44})\nabla_b^2 W + \omega k h_{14}\nabla_b^2 C_1 - ik\frac{z_{14}}{\mu_{11}}\left(kT - \frac{\omega}{\gamma^2 c^2}\nabla_b^2 M\right) = 0,$$

$$\nabla_b^2 B + 2\Delta c_{66}(\partial_{yy}-\partial_{xx})\psi_1 - k^2\frac{z_{14}}{\mu_{11}}\nabla_b^2 C_1 + ikh_{14}\left(-\omega T - \frac{k}{\gamma^2}\nabla_b^2 M\right) = 0,$$

$$ik(c_{13}+f_{44})\nabla_b^2\varphi + (f_{44}\nabla_b^2 + \varrho\omega^2 - c_{33}k^2)W - i\omega h_{14}\nabla_b^2 C_1 - \frac{z_{14}}{\mu_{11}}\left(kT - \frac{\omega}{\gamma^2 c^2}\nabla_b^2 M\right) = 0,$$

$$\nabla_b^2 C_1 - Z^2 C_1 - i\omega\bar{\mu}_{11}\bar{h}_{14}(ik\varphi + W) - k^2 z_{14}\psi - i\frac{\bar{\mu}_{11}}{c}\Lambda M = 0,$$

$$-\varepsilon_1\nabla_b^2 M + \varepsilon_3\bar{Z}^2 M - ik^2\bar{h}_{14}\nabla_b^2\psi - iZ^2 c\Lambda\nabla_b^2 C_1 - z_{14}\omega\varepsilon_1\nabla_b^2(ik\varphi + W) = 0, \tag{11}$$

where

$$A = (c_{11}\nabla_b^2 + \varrho\omega^2 - f_{44}k^2)\varphi,$$

$$B = \left(\frac{c_{11}-c_{12}}{2}\nabla_b^2 + \varrho\omega^2 - f_{44}k^2\right)\psi,$$

$$\psi_1 = (\partial_{yy}-\partial_{xx})\psi + 2\partial_{xy}\varphi, \tag{12}$$

$$f_{44} = c_{44} + \frac{z_{14}^2}{\mu_{11}}, \quad h_{14} = e_{14} - \frac{z_{14}\lambda_{11}}{\mu_{11}},$$

$$\Delta c_{66} = c_{66} - \frac{c_{11}-c_{12}}{2}, \quad \nabla_b^2 = \partial_{xx} + \partial_{yy},$$

$$\bar{\varepsilon}_{ij} = \frac{\varepsilon_{ij}}{\varepsilon_0}, \quad \lambda_i = \lambda_{ii} c,$$

$$\varepsilon_i = \bar{\varepsilon}_{ii} - \frac{\lambda_i^2}{\bar{\mu}_{ii}}, \quad \bar{h}_{14} = \frac{h_{14}}{\varepsilon_0},$$

$$\beta = \frac{\omega}{c}, \quad \bar{Z}^2 = k^2 - \beta^2 \varepsilon_1 \bar{\mu}_{11},$$

$$\Lambda = \frac{\lambda_3}{\bar{\mu}_{33}} - \frac{\lambda_1}{\bar{\mu}_{11}}, \quad \gamma^2 = \beta^2 - k^2, \tag{13}$$

and

$$T = \left(\frac{1}{\gamma^2}\nabla_b^2 + 1\right) G = = -\frac{\bar{\mu}_{11}}{Z^2}\nabla_b^2\left[\frac{\omega k}{c^2\gamma^2}\left(\varepsilon_1 - \frac{1}{\bar{\mu}_{11}}\right) M - ik\frac{\omega}{c^2}\bar{h}_{14}\psi - k\frac{z_{14}}{\bar{\mu}_{11}}(ik\varphi + W)\right]. \tag{14}$$

The symmetry 622 can be handled by setting $\Delta c_{66}=0$ in Eqs. (11)–(14). For the symmetry $4'22'$ the following equations are obtained:

$$2\partial_{xy}A - (\partial_{xx}-\partial_{yy})B + \Delta f_{66}\nabla_b^2\psi_1 + 2ik(c_{13}+f_{44})\partial_{xy}W + i\left(h_{36} - \frac{k^2}{\gamma^2}h_{14}\right)\nabla_b^2 M + ik\omega h_{14}\left(\frac{1}{\gamma^2}\nabla_b^2 + 1\right) G + \nabla_b^2\left(\frac{z_{36}}{\mu_{33}}\nabla_b^2 + k^2\frac{z_{14}}{\mu_{11}}\right) C_1 = 0, \tag{15}$$

$$(\partial_{xx}-\partial_{yy})A + 2\partial_{xy}B - k\omega h_{14}\nabla_b^2 C_1 + ik(c_{13}+f_{44})(\partial_{xx}-\partial_{yy})W + ik\frac{z_{14}}{\mu_{11}}\left\{k\left(\frac{1}{\gamma^2}\nabla_b^2 + 1\right) G - \frac{\omega}{c^2\gamma^2}\nabla_b^2 M\right\} = 0, \tag{16}$$

$$ik(c_{13}+f_{44})\nabla_b^2\varphi + (f_{44}\nabla_b^2 + \varrho\omega^2 - c_{33}k^2)W - 2ik\frac{z_{14}}{\mu_{11}}\partial_{xy}C_1 - h_{14}\left[-i\omega(\partial_{xx}-\partial_{yy})C_1 + 2\partial_{xy}\left\{i\omega H - \frac{1}{\gamma^2}(\omega G - kM)\right\}\right] - \frac{z_{14}}{\mu_{11}}\left[-(\partial_{xx}-\partial_{yy})\left\{\frac{1}{\gamma^2}\left(kG - \frac{\omega}{c^2}M\right) - ikH\right\}\right] = 0, \tag{17}$$

$$\nabla_b^2\left[\nabla_b^2 C_1 - Z_2^2 C_1 + z_{36}\frac{\mu_{11}}{\mu_{33}}\psi_1 - i\left(\lambda_{33}\frac{\mu_{11}}{\mu_{33}} - \lambda_{11}\right) M\right] - i\frac{\omega}{c^2}\bar{\mu}_{11}\bar{h}_{14}\{-ik\varphi_1 - (\partial_{xx}-\partial_{yy})W\} - ikz_{14}(ik\psi_1 + 2\partial_{xy}W) = 0, \tag{18}$$

$$-\varepsilon_1\nabla_b^2 M + \varepsilon_3\bar{Z}^2 M + i\bar{Z}^2 h_{36}\psi_1 + \bar{h}_{14}k(ik\psi_1 + 2\partial_{xy}W) - i\bar{Z}^2 c\Lambda\nabla_b^2 C_1 + z_{14}\omega\varepsilon_1\{ik\varphi_1 + (\partial_{xx}-\partial_{yy})W\} = 0, \tag{19}$$

with the additional condition (A6) of Appendix A and

$$\nabla_b^2 H - iG = 0 \tag{20}$$

and the further definitions

$$f_{66} = c_{66} + \frac{z_{36}^2}{\mu_{33}}, \quad h_{36} = e_{36} - \frac{z_{36}\lambda_{33}}{\mu_{33}},$$

$$\Delta f_{66} = f_{66} - \frac{c_{11}-c_{12}}{2}. \tag{21}$$

Equations (11)–(14) allow us to completely solve the problem for symmetries (a) and (b), whereas case (c) is solved from Eqs. (15)–(21) and (A6). In all the cases, the procedure is to write linear equations for φ, ψ, W, M, and C_1, after the calculation of G and H in terms of φ, ψ, W, and M through the additional conditions.

It is noteworthy to mention that the main physical issue that can be devised from the complex equations derived above is that the elastic constants are renormalized by the piezomagnetic coefficients and the magnetic susceptibilities; the piezoelectric coefficients are renormalized by the piezomagnetic and the magnetoelectric coefficients and the magnetic permeabilities, and finally the dielectric constants are renormalized by the magnetoelectric coefficients and the magnetic susceptibilities. It occurs that the elastic constants are positive, the piezoelectric and the piezomagnetic coefficient can be positive or negative, and the renormalized dielectric constants are positive only if $\varepsilon_{11}\mu_{11} > \lambda_{11}^2$ and $\varepsilon_{33}\mu_{33} > \lambda_{33}^2$. Except for the hexagonal symmetry, there are the operators ∂_{xy} and $\partial_{xx}-\partial_{yy}$ that break the cylindrical symmetry of the equations.

III. CALCULATION METHODS FOR THE MODES IN THE INFINITE MEDIUM

A. Linear equations

The main goal of this work is to calculate the frequencies of the acoustic and the electromagnetic modes in the infinite medium and to study their features upon changing the piezoelectric, the piezomagnetic, and the magnetoelectric parameters and the dielectric and the magnetic susceptibilities. It is convenient to work in the two-dimensional Fourier transform and to introduce adimensional quantities. It occurs that $(\partial_{xx}-\partial_{yy})f(x,y)$ and $2\partial_{xy}f(x,y)$ become $-q^2\tilde{f}(q_x,q_y)\cos 2\theta$ and $-q^2\tilde{f}(q_x,q_y)\sin 2\theta$, respectively, with $q_x = q\cos\theta$, $q_y = q\sin\theta$. Defining $Q=(q^2+k^2)^{1/2}$ it is possible to introduce the adimensional quantities

$$\tilde{c}_{ij} = \frac{c_{ij}}{c_{44}}, \quad \tilde{f}_{ij} = \frac{f_{ij}}{c_{44}},$$

$$E_{ij} = \frac{e_{ij}}{\sqrt{c_{44}\varepsilon_0}}, \quad Z_{ij} = \frac{z_{ij}}{\sqrt{c_{44}\mu_0}},$$

$$H_{14}=E_{14}-\frac{Z_{14}\lambda_1}{\bar{\mu}_{11}},\quad H_{36}=E_{36}-\frac{Z_{36}\lambda_3}{\bar{\mu}_{33}},\tag{22}$$

$k_1=\frac{k}{Q}$, $q_1=\frac{q}{Q}$, and the quantities

$$Y_1=Q^2\sqrt{c_{44}}\tilde{\varphi},\quad Y_2=iQ\sqrt{c_{44}}\tilde{W},$$

$$Y_3=i\sqrt{\varepsilon_0}\tilde{M},\quad Y_4=Q^2\sqrt{c_{44}}\tilde{\psi},$$

$$Y_5=Q^2\sqrt{\frac{1}{\mu_0}}\tilde{C}_1,\quad \alpha=\frac{1}{c}\sqrt{\frac{\varrho}{c_{44}}},$$

$$x=\sqrt{\frac{\varrho}{c_{44}}}\omega,\quad \beta=\frac{\omega}{c}=\alpha x,$$

$$y=\frac{x}{Q},\quad \frac{\beta}{Q}=\alpha y,\tag{23}$$

so that both sets of Eqs. (11)–(14) and Eqs. (15)–(21) and (A6) give two linear sets of five equations

$$\sum_{j=1}^{5}A_{ij}Y_j=0,\quad i=1,\ldots,5.\tag{24}$$

Apparently the matrix elements are different for the three symmetries studied. In particular, they coincide in cases (a) and (b), apart the additional condition $\Delta c_{66}=0$ for case (b). The matrix elements are called B_{ij} for cases (a) and (b) and A_{ij} for case (c). Being $Z^2=k_1^2-\alpha^2y^2\bar{\mu}_{11}\varepsilon_1$, we have

$$B_{11}=(\tilde{c}_{11}+\Delta\tilde{c}_{66}\sin^2 2\theta)q_1^2+\tilde{f}_{44}k_1^2-\frac{k_1^4Z_{14}^2}{Z^2\bar{\mu}_{11}}-y^2,$$

$$B_{12}=-k_1\left(\tilde{c}_{13}+\tilde{f}_{44}-\frac{k_1^2Z_{14}^2}{Z^2\bar{\mu}_{11}}\right),$$

$$B_{13}=\frac{k_1}{Z^2}Z_{14}\varepsilon_1\alpha y,$$

$$B_{14}=-q_1^2\Delta\tilde{c}_{66}\sin 2\theta\cos 2\theta-\frac{k_1^2}{\bar{\mu}_{11}}Z_{14}H_{14}\alpha y,$$

$$B_{15}=-k_1H_{14}\alpha y,\tag{25}$$

$$B_{21}=-q_1^2\Delta\tilde{c}_{66}\sin 2\theta\cos 2\theta-\frac{k_1^3}{Z^2}Z_{14}H_{14}\alpha y,$$

$$B_{22}=\frac{k_1^2}{Z^2}Z_{14}H_{14}\alpha y,$$

$$B_{23}=\frac{k_1^2}{Z^2}H_{14},$$

$$B_{24}=\left(\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}+\Delta\tilde{c}_{66}\cos^2 2\theta\right)q_1^2+\tilde{f}_{44}k_1^2-\frac{k_1^2}{Z^2}\bar{\mu}_{11}H_{14}^2\alpha^2y^2-y^2,$$

$$B_{25}=\frac{k_1^2}{\bar{\mu}_{11}}Z_{14},\tag{26}$$

$$B_{31}=k_1q_1^2\left(\tilde{c}_{11}+\tilde{f}_{44}-\frac{k_1^2Z_{14}^2}{Z^2\bar{\mu}_{11}}\right),$$

$$B_{32}=y^2-q_1^2\left(\tilde{f}_{44}-\frac{k_1^2Z_{14}^2}{Z^2\bar{\mu}_{11}}\right)-\tilde{c}_{33}k_1^2,$$

$$B_{33}=\frac{q_1^2}{Z^2}Z_{14}\varepsilon_1\alpha y,$$

$$B_{34}=-\frac{q_1^2k_1^2}{Z^2}Z_{14}H_{14}\alpha y,$$

$$B_{35}=-q_1^2H_{14}\alpha y,\tag{27}$$

$$B_{41}=k_1\bar{\mu}_{11}H_{14}\alpha y,$$

$$B_{42}=-\bar{\mu}_{11}H_{14}\alpha y,$$

$$B_{43}=-\bar{\mu}_{11}\Lambda,$$

$$B_{44}=-k_1^2Z_{14},$$

$$B_{45}=-(q_1^2+k_1^2-\varepsilon_1\bar{\mu}_{11}\alpha^2y^2),\tag{28}$$

$$B_{51}=-k_1q_1^2\varepsilon_1Z_{14}\alpha y,$$

$$B_{52}=q_1^2\varepsilon_1Z_{14}\alpha y,$$

$$B_{53}=q_1^2\varepsilon_1+\varepsilon_3Z^2,$$

$$B_{54}=-k_1^2q_1^2H_{14},$$

$$B_{55}=-Z^2q_1^2\Lambda.\tag{29}$$

To find the matrix elements for case (c), it is convenient to define the quantities

$$\alpha_1=y^2-\tilde{f}_{44}k_1^2-\left(\tilde{f}_{66}+\frac{\tilde{c}_{11}+\tilde{c}_{12}}{2}\right)q_1^2,$$

$$\alpha_2=y^2-\tilde{f}_{44}k_1^2-\tilde{c}_{11}q_1^2,$$

$$\alpha_3=y^2-\tilde{f}_{44}q_1^2-\tilde{c}_{33}k_1^2,$$

$$\beta_1=y^2-\tilde{f}_{44}k_1^2-\tilde{f}_{66}q_1^2,$$

$$\beta_2 = y^2 - \tilde{f}_{44}k_1^2 - \frac{\tilde{c}_{11} - \tilde{c}_{12}}{2}q_1^2,$$

$$H_1 = -\alpha y H_{14}\sin 2\theta - \frac{k_1 Z_{14}}{\bar{\mu}_{11}}\cos 2\theta,$$

$$G_1 = \alpha y H_{14}\cos 2\theta - \frac{k_1 Z_{14}}{\bar{\mu}_{11}}\sin 2\theta, \tag{30}$$

so that

$$A_{11} = -\alpha_1 \sin 2\theta + \frac{\bar{\mu}_{11}}{Z^2}H_{14}H_1k_1^2\alpha y,$$

$$A_{12} = -k_1\left[(\tilde{c}_{13} + \tilde{f}_{44})\sin 2\theta + \frac{\bar{\mu}_{11}}{Z^2}H_{14}H_1\alpha y\right],$$

$$A_{13} = -\left(H_{36} + \frac{k_1^2}{Z^2}H_{14}\right),$$

$$A_{14} = \beta_1 \cos 2\theta + \frac{\bar{\mu}_{11}}{Z^2}H_{14}G_1k_1^2\alpha y,$$

$$A_{15} = \frac{Z_{36}}{\bar{\mu}_{33}}q_1^2 - \frac{Z_{14}}{\bar{\mu}_{11}}k_1^2, \tag{31}$$

$$A_{21} = -\alpha_2 \cos 2\theta + Z_{14}H_1\frac{k_1^3}{Z^2},$$

$$A_{22} = -k_1\left\{(\tilde{c}_{13} + \tilde{f}_{44})\cos 2\theta + Z_{14}H_1\frac{k_1}{Z^2}\right\},$$

$$A_{23} = -Z_{14}\alpha y \varepsilon_1\frac{k_1}{Z^2},$$

$$A_{24} = -\beta_2 \sin 2\theta + Z_{14}G_1\frac{k_1^3}{Z^2},$$

$$A_{25} = \alpha y k_1 H_{14}, \tag{32}$$

$$A_{31} = k_1q_1^2\left[\tilde{c}_{13} + \tilde{f}_{44}H_1\frac{\bar{\mu}_{11}}{Z^2}\left(\alpha y H_{14}\sin 2\theta + k_1\frac{Z_{14}}{\bar{\mu}_{11}}\cos 2\theta\right)\right],$$

$$A_{32} = \alpha_3 - q_1^2H_1\frac{\bar{\mu}_{11}}{Z^2}\left(\alpha y H_{14}\sin 2\theta + k_1\frac{Z_{14}}{\bar{\mu}_{11}}\cos 2\theta\right),$$

$$A_{33} = \frac{q_1^2}{Z^2}(-k_1H_{14}\sin 2\theta - \alpha y Z_{14}\varepsilon_1 \cos 2\theta),$$

$$A_{34} = k_1q_1^2G_1\frac{\bar{\mu}_{11}}{Z^2}\left(\alpha y H_{14}\sin 2\theta + k_1\frac{Z_{14}}{\bar{\mu}_{11}}\cos 2\theta\right),$$

$$A_{35} = -q_1^2\left(-\alpha y H_{14}\cos 2\theta + k_1\frac{Z_{14}}{\bar{\mu}_{11}}\sin 2\theta\right), \tag{33}$$

$$A_{41} = \bar{\mu}_{11}\left\{\left(\frac{Z_{36}}{\bar{\mu}_{33}}q_1^2 - k_1^2\frac{Z_{14}}{\bar{\mu}_{11}}\right)\sin 2\theta + \alpha y k_1 H_{14}\cos 2\theta\right\},$$

$$A_{42} = \bar{\mu}_{11}\left(-\alpha y H_{14}\cos 2\theta + k_1\frac{Z_{14}}{\bar{\mu}_{11}}\sin 2\theta\right),$$

$$A_{43} = \bar{\mu}_{11}\Lambda,$$

$$A_{44} = \bar{\mu}_{11}\left\{\left(-\frac{Z_{36}}{\bar{\mu}_{33}}q_1^2 + k_1^2\frac{Z_{14}}{\bar{\mu}_{11}}\right)\cos 2\theta + \alpha y k_1 H_{14}\sin 2\theta\right\},$$

$$A_{45} = -(\alpha^2y^2\bar{\mu}_{11}\varepsilon_1 - 1), \tag{34}$$

$$A_{51} = q_1^2[\{(H_{14} + H_{36})k_1^2 - H_{36}\alpha^2y^2\bar{\mu}_{11}\varepsilon_1\}\sin 2\theta + Z_{14}\alpha y k_1\varepsilon_1 \cos 2\theta],$$

$$A_{52} = -q_1^2(H_{14}k_1 \sin 2\theta + Z_{14}\alpha y\varepsilon_1 \cos 2\theta),$$

$$A_{53} = \varepsilon_1q_1^2 + \varepsilon_3k_1^2 - \alpha^2y^2\varepsilon_1\varepsilon_3\bar{\mu}_{11},$$

$$A_{54} = -q_1^2[\{(H_{14} + H_{36})k_1^2 - H_{36}\alpha^2y^2\bar{\mu}_{11}\varepsilon_1\}\cos 2\theta - Z_{14}\alpha y k_1\varepsilon_1 \sin 2\theta],$$

$$A_{55} = -q_1^2(k_1^2 - \alpha^2y^2\varepsilon_1\bar{\mu}_{11})\Lambda. \tag{35}$$

The frequencies of the modes are given by the zeros of the determinants $\|A_{ij}\|$ or $\|B_{ij}\|$ that depend on several types of couplings, for example, $H_{ij}H_{mn}$ (renormalized piezoelectric coupling), $Z_{ij}Z_{mn}$ (piezomagnetic coupling), $\lambda_i\lambda_j$ (magnetoelectric coupling), $H_{ij}Z_{mn}$ (piezoelectric-piezomagnetic coupling), $H_{ij}\lambda_m$ (piezoelectric-magnetoelectric coupling), $Z_{ij}\lambda_m$ (piezomagnetic-magnetoelectric coupling), $H_{ij}Z_{mn}\lambda_p$ (piezoelectric-piezomagnetic-magnetoelectric coupling).

B. Methods of solutions

For crystals with the symmetries 422 and 622, the number of the parameters is strongly reduced in comparison with case (c), so that the discussion of the results is simplified. In any case, two main issues come out in the calculation and the interpretation of the solutions: the first one is that the value of the determinant becomes huge for high frequencies, so that it is very difficult to find the very narrow range where it becomes zero. The second one is that there are many parameters that can be changed. A method is proposed that allows us to calculate with accuracy and separately the acoustical and the electromagnetic eigenfrequencies.

Since the parameter α is the ratio between the sound and the light velocities, its value is very small (around 10^{-5}), so that $\alpha y \ll 1$ if $y \simeq 1$, whereas $y \gg 1$ if $\alpha y \simeq 1$. These two regimes occur when the acoustic and the electromagnetic modes are studied. In the first case, by developing the matrix elements A_{ij} or B_{ij} up to the second order in αy, it is obtained, for example, that

$$\sum_{p=0}^{2} A_{ij}^{(p)}(\alpha y)^p, \quad (36)$$

so that the determinant $\|A_{ij}\|=D$ is written as

$$D = D_0 + \alpha y \sum_{i=1}^{5} D_i + (\alpha y)^2 \left[\sum_{i=1}^{5} \bar{D}_i + \sum_{i=1,\ldots,5}^{j=1,\ldots,5} D_{ij} \right], \quad (37)$$

where D_0 is the determinant of the matrix whose elements are $A_{ij}^{(0)}$; D_i and $\bar{D}_i$ are the determinants such that all elements are $A_{ps}^{(0)}$ except those of column i that are $A_{pi}^{(1)}$ and $A_{pi}^{(2)}$ ($p=1,\ldots,5$), respectively; and, finally, the determinants D_{ij} have all elements $A_{ps}^{(0)}$ except those of columns i and j that are $A_{pi}^{(1)}$ and $A_{pj}^{(1)}$ ($p=1,\ldots,5$). In Appendix B the terms $A_{ij}^{(p)}$ and the analogous $B_{ij}^{(p)}$ are written. The determinants D_0 are third order in the variable y^2 (see Appendix B) and are written for the tetragonal and the hexagonal symmetries. They depend on the renormalized elastic, piezoelectric, piezomagnetic, and magnetoelectric coefficients. It seems that even to the zeroth order the acoustic frequencies are modified by the mentioned parameters. In Appendix D and in the next section it is shown that there are cases when this does not occur. A second question is to know when the correction to the zeroth-order frequencies is of first or second order in the parameter α. When $\Sigma_{i=1}^{5} D_i=0$, the corrections are of the second order. Some specific cases in which first- or second-order corrections show up are discussed in Appendix D.

The electromagnetic modes are such that $\alpha y \simeq 1$ and consequently $y \simeq \frac{1}{\alpha} \gg 1$. By inspection, only the elements B_{11}, B_{24}, and B_{32} in the symmetries 422 and 622 and A_{11}, A_{14}, A_{21}, A_{24}, and A_{32} in the symmetry $4'22'$ are of the order α^{-2}, so that by developing the determinant with the Laplace rule, terms of the orders α^{-6}, α^{-4}, α^{-2}, α^{0}, α^{2}, and α^{4} are found. The coefficient of the term α^{-6} gives in all cases the same algebraic equation of second order in $\frac{\beta^2}{Q^2}$.

IV. DISCUSSION OF THE RESULTS

In Appendix C the case of the propagation of the acoustic and the electromagnetic signals along the x, the y, or the z direction is discussed. It is found that all the renormalization effects on the elastic and the piezoelectric coefficients disappear, but not the modification of the dielectric constants due to the magnetoelectric contribution. The piezoelectric and the piezomagnetic effects appear to the second order in α. The explicit calculations require the knowledge of the adimensional piezoelectric, piezomagnetic, and magnetoelectric coefficients E_{ij}, Z_{ij}, and λ_i, respectively: the first ones are well known for many classes of materials, so that the values used in this work are reliable; the second ones are less known, but there are some experimental data giving the order of magnitude.[28] Finally, the quantities λ_i are fixed by taking into account that the inequalities $\bar{\varepsilon}_{11}\bar{\mu}_{11} > \lambda_1^2$ and $\bar{\varepsilon}_{33}\bar{\mu}_{33} > \lambda_3^2$ must hold. Apart from the simple case presented in Appendix C, the other cases are discussed considering separately the electromagnetic and the acoustic regimes.

A. Electromagnetic modes

For all the symmetries considered, the electromagnetic frequencies are given to the zero order by

$$\frac{\beta_{(0)}^2}{Q^2} = \frac{\bar{b} \pm q_1^2\sqrt{\bar{c}}}{2\varepsilon_1\varepsilon_3\bar{\mu}_{11}},$$

$$\bar{b} = \varepsilon_1 q_1^2 + \varepsilon_3(k_1^2+1) + \bar{\mu}_{11}\Lambda^2 q_1^2,$$

$$\bar{c} = (\varepsilon_1 - \varepsilon_3)^2 + \bar{\mu}_{11}\Lambda^2[2(\varepsilon_1+\varepsilon_3) + \bar{\mu}_{11}\Lambda^2], \quad (38)$$

and to the first order by

$$\beta_{(1)}^2 = \beta_{(0)}^2 - \frac{\alpha^2 Q^4}{\pm q^2 \beta_{(0)}^2 \sqrt{\bar{c}}} T, \quad (39)$$

where T has a very complicated form and it is different in all the cases considered. The expression of T is not written for brevity. The main feature of Eq. (38) is that the zeroth-order frequencies depends only on the renormalized dielectric constant ε_1 and ε_3, the magnetic susceptibilities $\bar{\mu}_{11}$ and $\bar{\mu}_{33}$, and the magnetoelectric coefficients λ_1 and λ_3. The stability of the modes requires that ε_1 and ε_3 are both positive. All the effects due to the direction of $\mathbf{q}$ in the x-y plane and the piezoelectric and the piezomagnetic coefficients are included in the coefficient T. The zeroth-order results become exact for materials that are only magnetoelectric. In all the other cases the piezoelectric and the piezomagnetic effects and the angular dependence appear only to the second order in α. The features can be summarized in the following way: (a) the square frequencies $\frac{\beta_{(0)}^2}{Q^2}$ are always positive; (b) if $\lambda_i=0$ ($i=1,3$) and for any values of $\bar{\varepsilon}_{11} \geq 1$ and $\bar{\varepsilon}_{33} \geq 1$, it is found that both the phase velocities $\frac{\omega}{k}$ and $\frac{\omega}{q}$ and group velocities of the modes $\frac{d\omega}{dk}$ and $\frac{d\omega}{dq}$ are less than c; and (c) if λ_1 and/or λ_3 are not zero, it is found that not only the phase velocity, but even the group velocity can be larger than c. For brevity, one case is shown [Figs. 1(a)–1(c)] that gives $\frac{\beta_{(0)}^2}{Q^2}$, $\frac{1}{c}\frac{d\omega}{dk}$, and $\frac{1}{c}\frac{d\omega}{dq}$, respectively, when $\bar{\varepsilon}_{11}=1$, $\bar{\varepsilon}_{33}=3$, $\lambda_1=0.5$, and $\lambda_3=1.5$. $\frac{\beta_{(0)}^2}{Q^2}$ and $\frac{1}{c}\frac{d\omega}{dk}$ are shown as functions of k_1 ($0 \leq k_1 \leq 1$) and $\frac{1}{c}\frac{d\omega}{dq}$ as a function of $q_1=\sqrt{1-k_1^2}$. It is seen that the group velocities $\frac{1}{c}\frac{d\omega}{dk}$ become larger than c for $k_1 \to 1$ and $\frac{1}{c}\frac{d\omega}{dq}$ for $q_1 \to 1$. Since the group velocity cannot be larger than c, it means that the dielectric and the magnetic susceptibilities and the magnetoelectric coefficients must verify some inequalities. Since

$$\frac{1}{c}\frac{d\omega}{dk} = \frac{Q}{\beta_{(0)}} \frac{k_1}{\varepsilon_1\bar{\mu}_{11}},$$

$$\frac{1}{c}\frac{d\omega}{dq} = \frac{Q}{\beta_{(0)}} \frac{\varepsilon_1+\varepsilon_3+\bar{\mu}_{11}\Lambda^2 \pm \sqrt{\bar{c}}}{2\varepsilon_1\varepsilon_3\bar{\mu}_{11}}, \quad (40)$$

it is found that $\frac{1}{c}\frac{d\omega}{dk}$ is less than 1 for any k and q for both the electromagnetic modes if

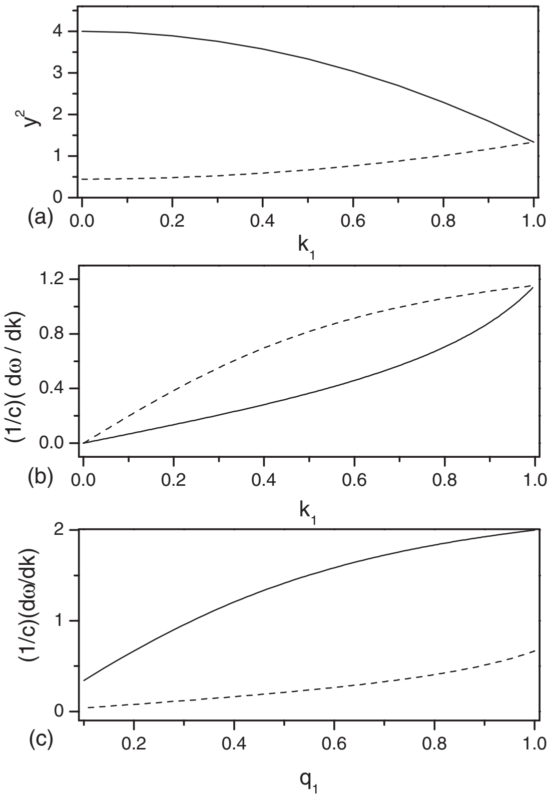

FIG. 1. (a) The square frequencies of the electromagnetic modes as a function of k_1 when $\bar{\mu}_{11}=\bar{\mu}_{33}=1$, $\bar{\varepsilon}_{11}=1$, $\bar{\varepsilon}_{33}=3$, $\lambda_1=0.5$, and $\lambda_3=1.5$. (b) The group velocity $\frac{1}{c}\frac{d\omega}{dk}$ of the electromagnetic modes as a function of k_1 when $\bar{\mu}_{11}=\bar{\mu}_{33}=1$, $\bar{\varepsilon}_{11}=1$, $\bar{\varepsilon}_{33}=3$, $\lambda_1=0.5$, and $\lambda_3=1.5$. (c) The group velocity $\frac{1}{c}\frac{d\omega}{dq}$ of the electromagnetic modes as function of q_1 when $\bar{\mu}_{11}=\bar{\mu}_{33}=1$, $\bar{\varepsilon}_{11}=1$, $\bar{\varepsilon}_{33}=3$, $\lambda_1=0.5$, and $\lambda_3=1.5$.

$$\bar{\varepsilon}_{11}\bar{\mu}_{11} \geq 1 + \lambda_1^2. \tag{41}$$

Furthermore, for the upper mode $\frac{1}{c}\frac{d\omega}{dq}$ satisfies the same condition if

$$2\varepsilon_1\varepsilon_3\bar{\mu}_{11} - \varepsilon_1 - \varepsilon_3 - \bar{\mu}_{11}\Lambda^2 \geq 0,$$

$$\sqrt{c} \leq 2\varepsilon_1\varepsilon_3\bar{\mu}_{11} - \varepsilon_1 - \varepsilon_3 - \bar{\mu}_{11}\Lambda^2. \tag{42}$$

The lower mode has the group velocity always less than c. If conditions (41) and (42) are not strictly verified, the modes can exist only in ranges of k and q.

B. Acoustic modes

To discuss the features of the acoustic modes, the following values of the elastic constants $\tilde{c}_{11}=2.0$, $\tilde{c}_{12}=0.12$, $\tilde{c}_{13}=0.21$, $\tilde{c}_{33}=1.85$, and $\tilde{c}_{66}=0.69$ are fixed and $k_1=0.5$. These values are consistent with those of the hexagonal piezoelectric ceramics.[5] The three acoustic frequencies when all piezoelectric, piezomagnetic, and magnetoelectric coefficients are zero are obtained starting from Eq. (B10) or Eq. (B20). There is a dependence on the angle θ that is more evident around $\theta=\frac{\pi}{4}$, because at $\theta=\frac{\pi}{4}$ the frequencies assume maximum or minimum values, and furthermore the curves show periodicity in the range $(0,\frac{\pi}{2})$. If $\Delta\tilde{c}_{66}=0$, the frequencies become independent of θ.

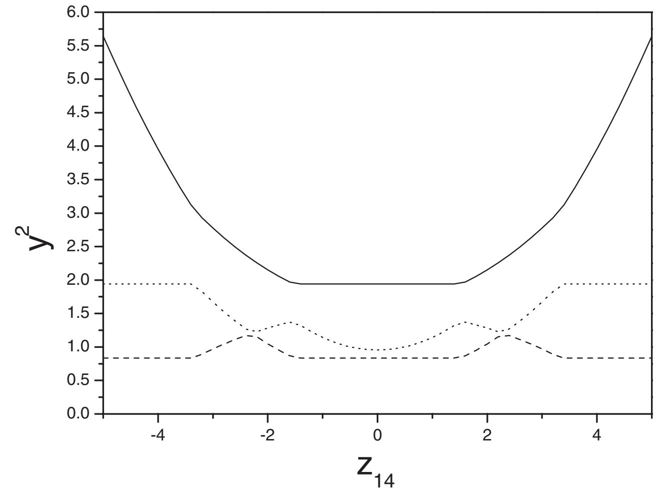

FIG. 2. The square frequencies of the acoustic modes in the tetragonal 422 symmetry as a function of Z_{14}, when $\theta=0$, $\bar{\mu}_{11}=\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, and $k_1=0.5$, $\lambda_1=\lambda_3=E_{14}=0$.

1. Acoustic modes in the hexagonal 622 and in the tetragonal 422 symmetries

These cases can be easily discussed looking to Eq. (B9) for the hexagonal symmetry and Eq. (B7) for the tetragonal 422 one. In the hexagonal case, two frequencies are purely elastic and the third shows a strong coupling to the piezoelectric, the piezomagnetic, and the magnetoelectric coefficients. It can be shown that also this last square frequency is always positive.

In the tetragonal symmetry 422 the angular dependence of the frequencies appears in the form $q^2\Delta\tilde{c}_{66}\sin 2\theta\cos 2\theta$ and the square of the frequencies are always positive, as it results from the numerical calculations. Apart from the angular dependence, the feature of the modes is that increasing $|Z_{14}|$ and/or $|E_{14}|$ the higher frequency increases, whereas the other two remain near to the acoustic ones. A unique feature that occurs is shown in Fig. 2, where the frequencies are shown as a function of Z_{14}, when $\theta=E_{14}=\lambda_1=\lambda_3=0$ and $\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=\bar{\mu}_{11}=\bar{\mu}_{33}=1$. There are ranges of Z_{14} where the frequencies are constant and coincident with the purely acoustic ones (in Fig. 2 this occurs for the lowest and the highest frequencies for Z_{14} around the zero mode and for the first and the second modes for higher Z_{14}). Furthermore, the detailed analysis of the modes around the crossing region of the two lowest frequencies shows that there is a true anticrossing rule. Similar results are obtained if $Z_{14}=0$ and $E_{14}\neq 0$, and the phenomenon occurs in a different way also if $\theta\neq 0$. In the cases described, the curves are symmetric if $Z_{14}\rightarrow -Z_{14}$ or $E_{14}\rightarrow -E_{14}$. On the other hand, if $E_{14}\neq 0$ and the frequencies are calculated again as a function of Z_{14}, the symmetry is lost and the higher mode is such that the two coefficients can give a cooperative contribution to increase the frequency; the same occurs if the magnetic susceptibility $\bar{\mu}_{11} > \bar{\mu}_{33}$. In this last case, the enhancement of the highest frequency occurs

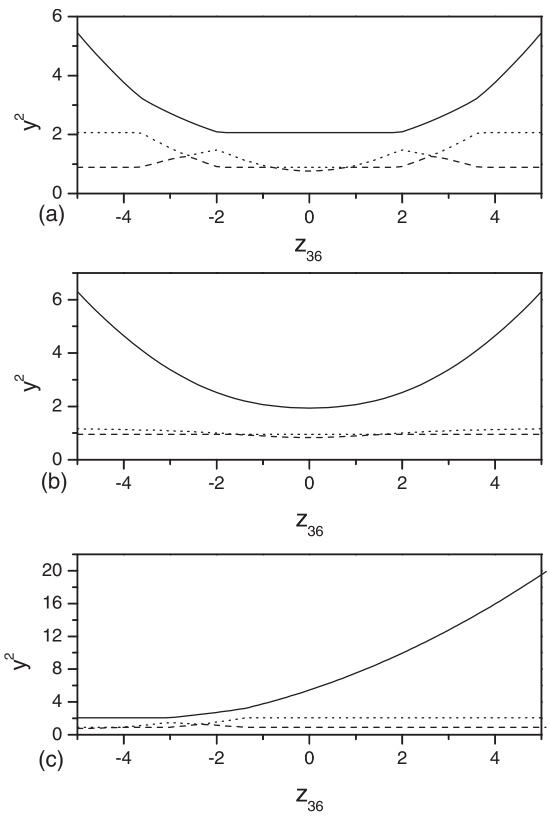

FIG. 3. (a) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=0$, $\bar{\mu}_{11}=\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=Z_{14}=0$. (b) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=\frac{\pi}{4}$, $\bar{\mu}_{11}=\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=Z_{14}=0$. (c) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=0$, $Z_{14}=5$, $\bar{\mu}_{11}=\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=0$.

for negative (positive) Z_{14} when E_{14} and λ_1 have the same (opposite) sign.

2. Acoustic modes in the tetragonal symmetry $4'22'$

The features of the acoustic modes are difficult to study, because there are many parameters that can change. For this reason it is convenient to study particular situations before considering the general case.

Piezomagnetic materials. If only the parameters Z_{36} and Z_{14} are not zero, the following features are obtained: (a) the dependence on the angle θ is similar to that obtained previously with a maximum or a minimum at $\theta=\frac{\pi}{4}$; (b) in Figs. 3(a) and 3(b) the dependence of the frequencies on Z_{36} is shown when $\theta=0$, $\frac{\pi}{4}$, $Z_{14}=E_{14}=E_{36}=0$, $\lambda_i=0$ $(i=1,3)$, and $\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=\bar{\mu}_{11}=\bar{\mu}_{33}=1$: the same phenomenon described for the previous tetragonal case is found, with the presence of two frequencies independent of Z_{36} in Fig. 3(a) and of only one in Fig. 3(b); (c) if $Z_{36}\leftrightarrow Z_{14}$, the same figures are obtained; and (d) if $Z_{14}\neq 0$, the symmetry of the curves when $Z_{36}\to -Z_{36}$ is lost, as it appears in Fig. 3(c), where $\theta=0$, $Z_{14}=5$, $E_{14}=E_{36}=0$, $\lambda_i=0$ $(i=1,3)$, and $\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=\bar{\mu}_{11}=\bar{\mu}_{33}=1$.

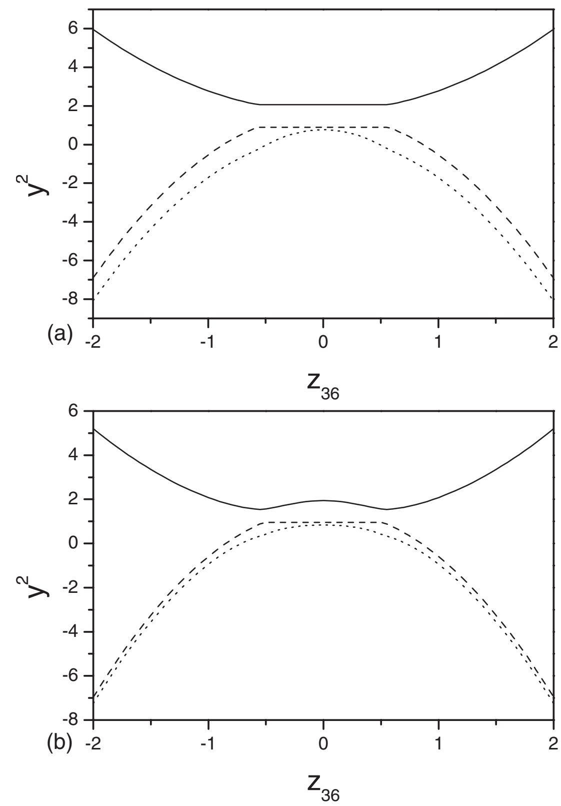

FIG. 4. (a) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=0$, $\bar{\mu}_{11}=7$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=Z_{14}=0$. (b) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=\frac{\pi}{4}$, $\bar{\mu}_{11}=7$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=Z_{14}=0$.

In Appendix D it is shown that one frequency can be easily calculated when $\sin 2\theta=0$ or $\cos 2\theta=0$ [see Eqs. (D4) and (D8)]. Such formulas show that in the first case the square frequencies are strongly modified by the piezomagnetic coefficient, whereas in the second one it is purely elastic apart from terms in α^2. Equation (D4) is particularly interesting because if $\bar{\mu}_{33}=\bar{\mu}_{11}$ or $Z_{36}=0$ or if q or k are zero, the square frequency is always positive; but, if k and q are not zero and $\frac{\bar{\mu}_{11}}{\bar{\mu}_{33}}$ and Z_{36} are sufficiently large, this quantity becomes negative. In Figs. 4(a) and 4(b) the acoustic frequencies are shown as a function of Z_{36} for $\theta=0,\frac{\pi}{4}$, when $Z_{14}=E_{14}=E_{36}=\lambda_i=0$ $(i=1,3)$, $\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=\bar{\mu}_{33}=1$, and $\bar{\mu}_{11}=7$. It is found that, for only a range of Z_{36} around zero, the three frequencies are all positive. Furthermore, it appears again as the phenomenon of the frequencies independent of Z_{36}. The increase in Z_{14} eliminates the symmetry of the curves for the transformation $Z_{36}\to -Z_{36}$, but the instability of the modes is only partially removed.

Finally, if $Z_{36}=0$ and Z_{14} is varied, the square frequencies are always positive, although there are ranges of Z_{14} where they are constant as in the previous cases. This case is not discussed for brevity.

Piezoelectric materials. If piezoelectric materials are considered, it is found from Eq. (D2) and (D6) of Appendix D, obtained for $\sin 2\theta=0$ and $\cos 2\theta=0$, respectively, that these square frequencies are always positive. The general feature found in this case is that, apart from the stability of the modes for all values of the parameters E_{14} and E_{36}, it occurs again that the frequencies are independent of the piezoelectric coefficients.

Pure magnetoelectric material. Taking $E_{ij}=Z_{ij}=0$, the linear set whose matrix element is given by Eqs. (31)–(35) simplifies significantly, because the acoustic waves decouple completely from the electromagnetic ones. The solutions can be found without approximations. The acoustic solutions are given by $A_{24}=0$ with $\tilde{\psi}\neq 0$, $\tilde{\varphi}=\tilde{W}=\tilde{M}=\tilde{G}=\tilde{C}_1=0$, $u=\partial_y\psi$ (i.e., $\tilde{u}=iq_y\tilde{\psi}$), $v=-\partial_x\psi$ (i.e., $\tilde{v}=-iq_x\tilde{\psi}$), $\partial_x u+\partial_y v=0$, and $A_{11}A_{32}-A_{12}A_{31}=0$ with $\tilde{\varphi}$ and $\tilde{W}$ not zero and $\tilde{\psi}=\tilde{M}=\tilde{G}=\tilde{C}_1=0$. The electric and the magnetic fields are zero and the frequencies are given by

$$x^2=k^2+\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}q^2,$$

$$x^2=\frac{b_1\pm\sqrt{b_1^2-4c_1}}{2},$$

$$b_1=q^2\left(1+\tilde{c}_{66}+\frac{\tilde{c}_{11}+\tilde{c}_{12}}{2}\right)+k^2(1+\tilde{c}_{33}),$$

$$c_1=\left[k^2+\left(\tilde{c}_{66}+\frac{\tilde{c}_{11}+\tilde{c}_{12}}{2}\right)q^2\right](q^2+\tilde{c}_{33}k^2)-q^2k^2(1+\tilde{c}_{13})^2. \quad (43)$$

The two electromagnetic modes are given by $A_{43}A_{55}-A_{53}A_{45}=0$, whose explicit solutions are given by Eq. (38). It occurs that $\tilde{M}$, $\tilde{C}_1$, and $\tilde{G}$ are not zero and $\tilde{\varphi}=\tilde{W}=\tilde{\psi}=0$ (i.e., $\tilde{u}=\tilde{v}=\tilde{W}=0$). No angular dependence is found.

General case. The final aim is to study if it is possible to eliminate the instability introduced by the piezomagnetic coefficient Z_{36} changing the piezoelectric and the magnetoelectric coefficients. The best results could be to write inequalities that should be verified to obtain positive square frequencies. Since the square frequencies are solutions of a third-order algebraic equation, the simple way to have such relation is to say that the square frequencies are positive, but these conditions contain not only the piezoelectric, the piezomagnetic, and the magnetoelectric coefficients, but also the wave numbers k and q and the angle θ. In our opinion, it should be interesting to find conditions between the structural coefficients that assure the square frequencies to be positive for any k and q, but this is difficult to be achieved. For this reasons the particular situation in which $Z_{36}=2$, $Z_{14}=5$, and $\bar{\mu}_{11}=7$ are taken as starting point, that, as seen in Figs. 5(a) and 5(b), gives two negative square frequencies. In Fig. 6(a) the frequencies are shown as a function of E_{36} when the other parameters are $\theta=0$, $E_{14}=0$, and $\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=\bar{\mu}_{33}=1$. Similar results are obtained for $\theta\neq 0$. It is found that negative value of E_{36} allows us to eliminate completely the instability. In Fig. 6(b) it is shown that also the magnetoelectric coefficients can contribute to eliminate the instability, as it is seen because the calculations are done with the same values of the parameters of Fig. 6(a), but taking $\lambda_1=2.5$. If the negative value of λ_1 is taken, the effect of the magnetoelectric coefficient on the highest frequency is much less important. The same occurs if the parameter λ_3 is taken as nonzero.

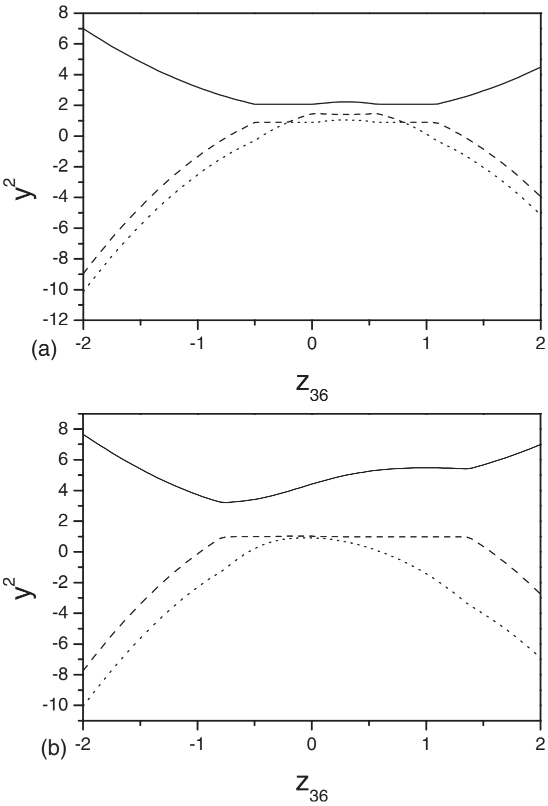

FIG. 5. (a) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=0$, $\bar{\mu}_{11}=7$, $Z_{14}=5$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=0$. (b) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of Z_{36}, when $\theta=\frac{\pi}{4}$, $\bar{\mu}_{11}=7$, $Z_{14}=5$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=E_{36}=0$.

V. CONCLUSIONS

In this work the problem of the propagation of coupled acoustic and electromagnetic signals has been solved in materials with hexagonal and two types of tetragonal spatial magnetic symmetry. The procedure to find the set of five independent equations, whose solutions give three acoustic and two electromagnetic modes, has been described in detail. Since the acoustic and the electromagnetic frequencies differ by many orders of magnitude, a procedure has been introduced that allows us to treat separately the acoustic and the electromagnetic cases, using the fact that the parameter α (the ratio between the sound and the vacuum light velocities) enters the theory.

The general results can be summarized as follows:

(a) the elastic constants of the material are renormalized by the piezomagnetic parameters and the magnetic susceptibilities;

(b) the piezoelectric constants are renormalized by the piezomagnetic and the magnetoelectric constants and the

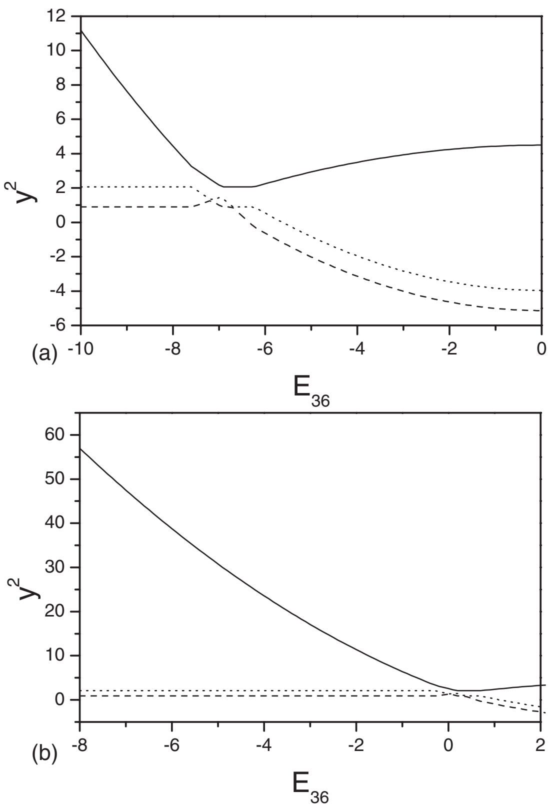

FIG. 6. (a) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of E_{36}, when $\theta=0$, $\bar{\mu}_{11}=7$, $Z_{14}=5$, $Z_{36}=2$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_1=\lambda_3=E_{14}=0$. (b) The square frequencies of the acoustic modes in the tetragonal $4'22'$ symmetry as a function of E_{36}, when $\theta=0$, $\bar{\mu}_{11}=7$, $Z_{14}=5$, $Z_{36}=2$, $\lambda_1=2.5$, $\bar{\mu}_{33}=\bar{\varepsilon}_{11}=\bar{\varepsilon}_{33}=1$, $k_1=0.5$, and $\lambda_3=E_{14}=0$.

magnetic susceptibility, whose contribution appears as a product of the type $\frac{z_{ij}}{\bar{\mu}_{ii}}\lambda_{ii}$;

(c) the dielectric constants are renormalized by the magnetoelectric coefficients and the magnetic susceptibilities with a contribution of the type $\frac{\lambda_{ii}^2}{\bar{\mu}_{ii}}$;

(d) the problem can be discussed using the set of adimensional quantities: $\tilde{c}_{ij}=\frac{c_{ij}}{c_{44}}$, $E_{ij}=\frac{e_{ij}}{\sqrt{c_{44}\varepsilon_0}}$, $Z_{ij}=\frac{z_{ij}}{\sqrt{c_{44}\mu_0}}$, $\lambda_i=\lambda_{ii}c$, $\bar{\varepsilon}_{ii}=\frac{\varepsilon_{ii}}{\varepsilon_0}$, and $\bar{\mu}_{ii}=\frac{\mu_{ii}}{\mu_0}$. In the calculations we have done, the values used for these quantities are reliable, although the piezomagnetic coefficients are known only for few materials;[28] and (e) the electromagnetic modes are given, to the zeroth order, by the same equation independent of the symmetry and of the angle θ, but depending on the renormalized dielectric constants and the magnetoelectric coefficients; the contribution of the piezoelectric and the piezomagnetic coefficients and the angle appears as a term of the second order in α. Furthermore the electromagnetic square frequencies are positive if $\bar{\varepsilon}_{11}\bar{\mu}_{11}>\lambda_1^2$ and $\bar{\varepsilon}_{33}\bar{\mu}_{33}>\lambda_3^2$ and the group velocity is less than c if the dielectric constants, the magnetic susceptibilities, and the magnetoelectric coefficients satisfy the inequalities given by Eqs. (41) and (42). Results specific of each studied symmetry are instead summarized separately in the following.

A. Hexagonal 622 symmetry

The acoustic modes are independent of θ and their frequencies can be calculated explicitly [see Eq. (B9)]. Two modes are purely acoustic and the third one shows a strong coupling to the piezoelectric and the piezomagnetic coefficients. The square frequency is always positive.

B. Tetragonal 422 symmetry

In this case the acoustic modes depend on θ through the term $(\tilde{c}_{66}-\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2})\sin 2\theta \cos 2\theta$. The acoustic modes have the peculiar feature that the highest frequency, for large values of Z_{14} and E_{14}, increases with a cooperative effect of the piezoelectric and the magnetoelectric coefficients. Furthermore, there is the possibility of finding frequencies independent of Z_{14} and/or E_{14}, coincident with the elastic frequencies (i.e., the renormalization contribution to the elastic and the piezoelectric constants disappears).

C. Tetragonal $4'22'$ symmetry

In this case all the features of the frequencies found in the previous case are still present, with a possibility of having negative square acoustic frequencies for reliable values of the coefficient Z_{36}. A particular situation is discussed in this work to eliminate the instabilities, because no general criterion of stability is found.

ACKNOWLEDGMENTS

We acknowledge the financial support from the National PON Project "Sonar Tridimensionale Stereoscopic Simultaneo a 500 kHz" (Contract No. STSS-500) and from the MIUR PRIN2007 project.

APPENDIX A

In this appendix we show the derivation of some equations of Sec. II. By substituting Eq. (8) in Eq. (7) the following are obtained:

$$\partial_x A + \partial_y B + \Delta c_{66}\partial_y \psi_1 + ik(c_{13}+f_{44})\partial_x W - h_{36}\partial_y E_z - ikh_{14}E_y - \frac{z_{36}}{\mu_{33}}\partial_y B_z - ik\frac{z_{14}}{\mu_{11}}B_y = 0,$$

$$\partial_y A - \partial_x B + \Delta f_{66}\partial_x \psi_1 + ik(c_{13}+f_{44})\partial_y W - h_{36}\partial_x E_z - ikh_{14}E_x - \frac{z_{36}}{\mu_{33}}\partial_x B_z - ik\frac{z_{14}}{\mu_{11}}B_x = 0,$$

$$ik(c_{13}+f_{44})\nabla_b^2\varphi + [f_{44}\nabla_b^2 + (\varrho\omega^2 - c_{33}k^2)] \times W - h_{14}(\partial_x E_y + \partial_y E_x) - \frac{z_{14}}{\mu_{11}}(\partial_x B_y + \partial_y B_x) = 0. \quad \text{(A1)}$$

If the first and the second of Eqs. (A1) are first derived with respect to y and x, respectively, and then summed and first derived with respect to x and y and then subtracted, the following are obtained:

$$2\partial_{xy}A + (\partial_{yy} - \partial_{xx})B + \Delta f_{66}\nabla_b^2\psi_1 + 2ik(c_{13} + f_{44})\partial_{xy}W + ih_{36}\nabla_b^2 M + ikh_{14}(\omega G + \nabla_b^2\phi) - \left(\frac{z_{36}}{\mu_{33}}\nabla_b^2 + k^2\frac{z_{14}}{\mu_{11}}\right)B_z = 0,$$

$$-(\partial_{yy} - \partial_{xx})A + 2\partial_{xy}B - ik(c_{13} + f_{44})(\partial_{yy} - \partial_{xx})W + k\omega h_{14}B_z + ik\frac{z_{14}}{\mu_{11}}(\nabla_b^2 F_z + kG) = 0, \quad \text{(A2)}$$

where M, G, $\mathbf{F}$, $\mathbf{F}_b$, C_1, and H are defined in Sec. II. For what concerns the equations for the scalar and the vector potentials, the sixth equation in Eq. (7) is identically satisfied if the fourth equation is used. For F_x, F_y, and F_z the following equations are obtained, respectively

$$\nabla_b^2 F_x - (k^2 - \beta^2\varepsilon_1)F_x - i\frac{\omega}{c^2}\left[-(\varepsilon_1 - 1)\partial_x\phi + \frac{\lambda_{11}c^2}{\bar{\mu}_{11}}B_x + \bar{h}_{14}\{ik(\partial_y\varphi - \partial_x\psi) + \partial_y W\}\right] + \left(1 - \frac{1}{\bar{\mu}_{33}}\right)\partial_y B_z + \frac{z_{36}}{\bar{\mu}_{33}}\partial_y\psi_1 + \frac{\lambda_{33}}{\bar{\mu}_{33}}\partial_y E_z - ik\left\{\left(1 - \frac{1}{\bar{\mu}_{11}}\right)B_y + \frac{\lambda_{11}}{\bar{\mu}_{11}}E_y\right\} - \frac{z_{14}}{\bar{\mu}_{11}}ik\{ik(\partial_x\varphi + \partial_y\psi) + \partial_x W\} = 0,$$

$$\nabla_b^2 F_y - (k^2 - \beta^2\varepsilon_1)F_y - i\frac{\omega}{c^2}\left[-(\varepsilon_1 - 1)\partial_y\phi + \frac{\lambda_{11}c^2}{\bar{\mu}_{11}}B_y + \bar{h}_{14}\{ik(\partial_x\varphi + \partial_y\psi) + \partial_x W\}\right] - \left[\left(1 - \frac{1}{\bar{\mu}_{33}}\right)\partial_x B_z + \frac{z_{36}}{\bar{\mu}_{33}}\partial_x\psi_1 + \frac{\lambda_{33}}{\bar{\mu}_{33}}\partial_x E_z\right] + ik\left\{\left(1 - \frac{1}{\bar{\mu}_{11}}\right)B_x + \frac{\lambda_{11}}{\bar{\mu}_{11}}E_x\right\} + \frac{z_{14}}{\bar{\mu}_{11}}ik\{ik(\partial_y\varphi - \partial_x\psi) + \partial_y W\} = 0,$$

$$\nabla_b^2 F_z + \bar{\mu}_{11}\gamma^2 F_z - \bar{\mu}_{11}\left(1 - \frac{1}{\bar{\mu}_{11}}\right)kG - i\bar{\mu}_{11}\frac{\omega}{c^2} \times \left[(\varepsilon_3 - 1)E_z + \bar{h}_{36}\psi_1 + \frac{c}{\mu_0}\Lambda B_z\right] + z_{14}\{ik\varphi_1 + (\partial_{xx} - \partial_{yy})W\} = 0, \quad \text{(A3)}$$

where

$$\varphi_1 = (\partial_{xx} - \partial_{yy})\varphi + 2\partial_{xy}\psi. \quad \text{(A4)}$$

By deriving the first and the second of Eqs. (A3) with respect to x and y, respectively, and then summing or deriving the same equations with respect to y and x, respectively, and then subtracting, we obtain

$$\nabla_b^2 G - \frac{1}{\bar{\mu}_{11}}Z^2 G + k\left(1 - \frac{1}{\bar{\mu}_{11}}\right)\nabla_b^2 F_z - \frac{\omega}{c^2}[-(\varepsilon_1 - 1)\nabla_b^2\phi + \bar{h}_{14}(ik\psi_1 + 2\partial_{xy}W)] - k\frac{z_{14}}{\bar{\mu}_{11}}\{ik\varphi_1 + (\partial_{xx} - \partial_{yy})W\} = 0,$$

$$\nabla_b^2\left[\nabla_b^2 C_1 - \bar{Z}^2 C_1 + \frac{z_{36}\bar{\mu}_{11}}{\bar{\mu}_{33}}\psi_1 - i\left(\lambda_{33}\frac{\mu_{11}}{\mu_{33}} - \lambda_{11}\right)M\right] - i\frac{\omega}{c^2}\bar{\mu}_{11}\bar{h}_{14}[-ik\varphi_1 - (\partial_{xx} - \partial_{yy})W] - ikz_{14}(ik\psi_1 + 2\partial_{xy}W) = 0, \quad \text{(A5)}$$

Since $\phi = \frac{1}{\gamma^2}(\omega G - kM)$ and $F_z = \frac{1}{\gamma^2}(kG - \frac{\omega}{c^2}M)$, from the first of Eqs. (A5), the third of Eqs. (A3), and the equation $\nabla\cdot\mathbf{D} = 0$ the following are obtained:

$$\frac{1}{\gamma^2}\nabla_b^2 G + G = -\frac{\bar{\mu}_{11}}{\bar{Z}^2}\left[\frac{\omega k}{c^2\gamma^2}\left(\varepsilon_1 - \frac{1}{\bar{\mu}_{11}}\right)\nabla_b^2 M + \frac{\bar{h}_{14}\omega}{c^2}(ik\psi_1 + 2\partial_{xy}W) + k\frac{z_{14}}{\bar{\mu}_{11}} \times \{ik\varphi_1 + (\partial_{xx} - \partial_{yy})W\}\right], \quad \text{(A6)}$$

$$\frac{1}{\gamma^2}\nabla_b^2 G + G = \frac{\omega}{kc^2}\left(\frac{1}{\gamma^2}\nabla_b^2 M + \varepsilon_3\bar{\mu}_{11}M\right) + i\frac{\omega\bar{\mu}_{11}}{kc^2}\left(\bar{h}_{36}\psi_1 + \frac{\Delta\lambda}{\varepsilon_0}B_z\right) - \frac{z_{14}}{k}\{ik\varphi_1 + (\partial_{xx} - \partial_{yy})W\}, \quad \text{(A7)}$$

$$\frac{1}{\gamma^2}\nabla_b^2 G + G = \frac{k}{\omega}\left(\frac{1}{\gamma^2}\nabla_b^2 M + \frac{\varepsilon_3}{\varepsilon_1}M\right)\frac{\bar{h}_{14}}{\varepsilon_1\omega}(ik\psi_1 + 2\partial_{xy}W) + i\frac{k}{\varepsilon_1\omega}(\bar{h}_{36}\psi_1 + c\Lambda B_z). \quad \text{(A8)}$$

By subtracting Eq. (A6) from Eq. (A7) and Eq. (A7) from Eq. (A8), Eq. (19) is obtained. Equations (15) and (16) are obtained from Eqs. (A2) with simple calculations; Eq. (17) it is obtained from the third of Eqs. (A1); finally, Eq. (18) is obtained from the second of Eqs. (A5).

APPENDIX B

This appendix shows the explicit form of the terms $A_{ij}^{(p)}$ and $B_{ij}^{(p)}$ introduced in Sec. III B. For cases (a) and (b), it is found that $B_{11}^{(1)} = B_{12}^{(1)} = B_{13}^{(0)} = B_{13}^{(2)} = B_{14}^{(2)} = B_{15}^{(0)} = B_{15}^{(2)} = 0$ and

$$B_{11}^{(0)} = (\tilde{c}_{11} + \Delta\tilde{c}_{66}\sin^2 2\theta)q_1^2 + k_1^2 - y^2,$$

$$B_{11}^{(2)} = -Z_{14}^2\varepsilon_1, \quad B_{12}^{(0)} = -k_1(\tilde{c}_{13} + 1),$$

$$B_{12}^{(2)} = Z_{14}^2\varepsilon_1, \quad B_{13}^{(1)} = \frac{Z_{14}}{k_1}\varepsilon_1,$$

$$B_{14}^{(0)}=-q_1^2\Delta\tilde{c}_{66}\sin 2\theta\cos 2\theta,$$

$$B_{14}^{(1)}=-\frac{k_1^2}{\bar{\mu}_{11}}Z_{14}H_{14},\quad B_{15}^{(1)}=-k_1H_{14}, \tag{B1}$$

$B_{21}^{(2)}=B_{22}^{(0)}=B_{22}^{(2)}=B_{23}^{(1)}=B_{24}^{(1)}=B_{25}^{(1)}=B_{25}^{(2)}=0$ and

$$B_{21}^{(0)}=-q_1^2\Delta\tilde{c}_{66}\sin 2\theta\cos 2\theta,\quad B_{21}^{(1)}=-k_1H_{14}Z_{14},$$

$$B_{22}^{(1)}=H_{14}Z_{14},\quad B_{23}^{(0)}=H_{14},\quad B_{23}^{(2)}=\frac{\bar{\mu}_{11}}{k_1^2}H_{14}\varepsilon_1,$$

$$B_{24}^{(0)}=\left(\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}+\Delta\tilde{c}_{66}\cos^2 2\theta\right)q_1^2+\tilde{f}_{44}k_1^2-y^2,$$

$$B_{24}^{(2)}=-\bar{\mu}_{11}H_{14}^2,\quad B_{25}^{(0)}=\frac{k_1^2Z_{14}}{\bar{\mu}_{11}}, \tag{B2}$$

$B_{31}^{(1)}=B_{32}^{(1)}=B_{33}^{(0)}=B_{33}^{(2)}=B_{34}^{(0)}=B_{34}^{(2)}=B_{35}^{(0)}=B_{35}^{(2)}=0$ and

$$B_{31}^{(0)}=k_1q_1^2(\tilde{c}_{11}+1),\quad B_{31}^{(2)}=-\frac{q_1^2}{k_1}Z_{14}^2\varepsilon_1,$$

$$B_{32}^{(0)}=y^2-q_1^2-c_{33}k_1^2,\quad B_{32}^{(2)}=\frac{q_1^2}{k_1^2}Z_{14}^2\varepsilon_1,$$

$$B_{33}^{(1)}=\frac{q_1^2}{k_1^2}Z_{14}\varepsilon_1,\quad B_{34}^{(1)}=-q_1^2H_{14}Z_{14},$$

$$B_{35}^{(1)}=-q_1^2H_{14}, \tag{B3}$$

$B_{41}^{(0)}=B_{41}^{(2)}=B_{42}^{(0)}=B_{42}^{(2)}=B_{43}^{(1)}=B_{43}^{(2)}=B_{44}^{(1)}=B_{44}^{(2)}=B_{45}^{(1)}=0$ and

$$B_{41}^{(1)}=k\bar{\mu}_{11}H_{14},\quad B_{42}^{(1)}=-\bar{\mu}_{11}H_{14},$$

$$B_{43}^{(0)}=-\bar{\mu}_{11}\Lambda,\quad B_{44}^{(0)}=-k^2Z_{14},$$

$$B_{45}^{(0)}=-(q^2+k^2),\quad B_{45}^{(2)}=\bar{\mu}_{11}\varepsilon_1, \tag{B4}$$

$B_{51}^{(0)}=B_{51}^{(2)}=B_{52}^{(0)}=B_{52}^{(2)}=B_{53}^{(1)}=B_{54}^{(1)}=B_{54}^{(2)}=B_{55}^{(1)}=0$, and

$$B_{51}^{(1)}=-k_1q_1^2\varepsilon_1Z_{14},\quad B_{52}^{(1)}=q_1^2\varepsilon_1Z_{14},$$

$$B_{53}^{(0)}=\varepsilon_1q_1^2+\varepsilon_3k_1^2,\quad B_{53}^{(2)}=-\varepsilon_1\varepsilon_3\bar{\mu}_{11},$$

$$B_{54}^{(0)}=-k_1^2q_1^2H_{14},\quad B_{55}^{(0)}=-k_1^2q_1^2\Lambda,$$

$$B_{55}^{(2)}=q_1^2\varepsilon_1\bar{\mu}_{11}\Lambda. \tag{B5}$$

To write the equation for the acoustic frequencies, it is convenient to define

$$b_{11}=(\tilde{c}_{11}+\Delta\tilde{c}_{66}\sin^2 2\theta)q_1^2+k_1^2,$$

$$b_{24}=\left(\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}+\Delta\tilde{c}_{66}\cos^2 2\theta\right)q_1^2+\tilde{f}_{44}k_1^2,$$

$$b_{32}=q_1^2+c_{33}k_1^2, \tag{B6}$$

and $b_{ij}=B_{ij}^{(0)}$ for the other indices. The equation that gives the acoustic frequencies in cases (a) and (b) is

$$(y^4-\eta_1y^2+\chi_1)(\eta_2y^2+\chi_2)-\eta_3y^2+\chi_3=0 \tag{B7}$$

with

$$\eta_1=b_{11}+b_{32},$$

$$\chi_1=b_{11}b_{32}+b_{31}b_{12},$$

$$\eta_2=b_{43}b_{55}-b_{53}b_{45},$$

$$\chi_2=b_{23}(b_{44}b_{55}-b_{54}b_{45})-b_{24}\eta_2+b_{25}(b_{43}b_{54}-b_{53}b_{44}),$$

$$\eta_3=b_{21}b_{14}(b_{43}b_{55}-b_{53}b_{45}),$$

$$\chi_3=b_{32}\eta_3. \tag{B8}$$

The form (B7) is convenient because in the case of the hexagonal group $\eta_3=\chi_3=0$ and the frequencies can be easily calculated. They are

$$y^2=\tfrac{1}{2}[(\tilde{c}_{11}+1)q_1^2+\tilde{c}_{33}k_1^2\pm\sqrt{\Delta}],$$

$$\Delta=[(\tilde{c}_{11}-1)q_1^2-\tilde{c}_{33}k_1^2]^2+4k_1^2q_1^2(\tilde{c}_{11}+1)(\tilde{c}_{13}+1),$$

$$y^2=\tilde{c}_{66}q_1^2+\tilde{f}_{44}k_1^2+k_1^2\frac{H_{14}^2q_1^2-\frac{Z_{14}^2}{\bar{\mu}_{11}}k_1^2(\varepsilon_1q_1^2+\varepsilon_3k_1^2)-2H_{14}Z_{14}\Lambda q_1^2k_1^2}{\bar{\mu}_{11}\Lambda^2q_1^2k_1^2+\varepsilon_1q_1^2+\varepsilon_3k_1^2}. \tag{B9}$$

The first two frequencies are purely acoustic, because the quantities η_1 and χ_1 do not depend on the piezoelectric or piezomagnetic coefficients, whereas the last one depends on the piezoelectric, the piezomagnetic, and the magnetoelectric coefficients. If $\Delta\tilde{c}_{66}\neq 0$, the third-order polynomial is

$$D_0=y^6+b_2y^4+b_1y^2+b_0 \tag{B10}$$

with

$$b_2=\frac{\chi_2-\eta_1\eta_2}{\eta_2},$$

$$b_1=\frac{\chi_1\eta_2-\eta_1\chi_2-\eta_3}{\eta_2},$$

$$b_0=\frac{\chi_1\chi_2+\chi_3}{\eta_2}, \tag{B11}$$

For case (c), by defining

$$H_1=H_{1,0}+\alpha yH_{1,1},\quad G_1=G_{1,0}+\alpha yG_{1,1},$$

$$H_{1,0}=-\frac{k_1Z_{14}}{\bar{\mu}_{11}}\cos 2\theta,\quad H_{1,1}=-H_{14}\sin 2\theta,$$

$$G_{1,0} = -\frac{k_1 Z_{14}}{\bar{\mu}_{11}}\sin 2\theta, \quad G_{1,1} = H_{14}\cos 2\theta, \qquad \text{(B12)}$$

the following are obtained:

$$A_{11}^{(0)} = -\alpha_1 \sin 2\theta, \quad A_{11}^{(1)} = \bar{\mu}_{11} H_{14} H_{1,0},$$

$$A_{11}^{(2)} = \bar{\mu}_{11} H_{14} H_{1,1},$$

$$A_{12}^{(0)} = -k_1(\tilde{c}_{13} + \tilde{f}_{44})\sin 2\theta,$$

$$A_{12}^{(1)} = -\frac{\bar{\mu}_{11}}{k_1} H_{14} H_{1,0}, \quad A_{12}^{(2)} = -\frac{\bar{\mu}_{11}}{k_1} H_{14} H_{1,1},$$

$$A_{13}^{(0)} = -(H_{14} + H_{36}), \quad A_{13}^{(1)} = 0,$$

$$A_{13}^{(2)} = -\frac{\bar{\mu}_{11} H_{14}\varepsilon_1}{k_1^2},$$

$$A_{14}^{(0)} = \beta_1 \cos 2\theta, \quad A_{14}^{(1)} = \bar{\mu}_{11} H_{14} G_{1,0},$$

$$A_{14}^{(2)} = \bar{\mu}_{11} H_{14} G_{1,1},$$

$$A_{15}^{(0)} = \frac{Z_{36}}{\bar{\mu}_{33}} q_1^2 - \frac{Z_{14}}{\bar{\mu}_{11}} k_1^2, \quad A_{15}^{(1)} = 0, \quad A_{15}^{(2)} = 0, \qquad \text{(B13)}$$

$$A_{21}^{(0)} = -\alpha_2 \cos 2\theta + k_1 Z_{14} H_{1,0},$$

$$A_{21}^{(1)} = k_1 Z_{14} H_{1,1}, \quad A_{21}^{(2)} = \frac{Z_{14} H_{1,0}\bar{\mu}_{11}\varepsilon_1}{k_1},$$

$$A_{22}^{(0)} = -k_1(\tilde{c}_{13} + \tilde{f}_{44})\cos 2\theta - Z_{14} H_{1,0},$$

$$A_{22}^{(1)} = -Z_{14} H_{1,1}, \quad A_{22}^{(2)} = -\frac{Z_{14} H_{1,0}\bar{\mu}_{11}\varepsilon_1}{k_1^2},$$

$$A_{23}^{(0)} = 0, \quad A_{23}^{(1)} = -\frac{Z_{14}\varepsilon_1}{k_1}, \quad A_{23}^{(2)} = 0,$$

$$A_{24}^{(0)} = -\beta_2 \sin 2\theta + k_1 Z_{14} G_{1,0},$$

$$A_{24}^{(1)} = k_1 Z_{14} G_{1,1}, \quad A_{24}^{(2)} = \frac{Z_{14} G_{1,0}\bar{\mu}_{11}\varepsilon_1}{k_1},$$

$$A_{25}^{(0)} = 0, \quad A_{25}^{(1)} = k_1 H_{14}, \quad A_{25}^{(2)} = 0, \qquad \text{(B14)}$$

$$A_{31}^{(0)} = q_1^2[k_1(\tilde{c}_{13} + \tilde{f}_{44}) + Z_{14} H_{1,0}\cos 2\theta],$$

$$A_{31}^{(1)} = q_1^2\left[\frac{\bar{\mu}_{11}}{k_1} H_{14} H_{1,0}\sin 2\theta + Z_{14} H_{1,1}\cos 2\theta\right],$$

$$A_{31}^{(2)} = q_1^2 \frac{\bar{\mu}_{11}}{k_1}\left(H_{14} H_{1,1}\sin 2\theta + \frac{Z_{14} H_{1,0}\varepsilon_1}{k_1}\cos 2\theta\right),$$

$$A_{32}^{(0)} = \alpha_3 - \frac{q_1^2}{k_1} Z_{14} H_{1,0}\cos 2\theta,$$

$$A_{32}^{(1)} = -\frac{q_1^2}{k_1^2}\bar{\mu}_{11}\left(H_{14} H_{1,0}\sin 2\theta + \frac{k_1}{\bar{\mu}_{11}} Z_{14} H_{1,1}\cos 2\theta\right),$$

$$A_{32}^{(2)} = -\frac{q_1^2}{k_1^2}\bar{\mu}_{11}\left(H_{14} H_{1,1}\sin 2\theta + \frac{Z_{14} H_{1,0}\varepsilon_1}{k_1}\cos 2\theta\right),$$

$$A_{33}^{(0)} = -\frac{q_1^2}{k_1} H_{14}\sin 2\theta, \quad A_{33}^{(1)} = -\frac{q_1^2}{k_1^2} Z_{14}\varepsilon_1 \cos 2\theta,$$

$$A_{33}^{(2)} = -\frac{q_1^2}{k_1^3}\bar{\mu}_{11} H_{14}\varepsilon_1 \sin 2\theta,$$

$$A_{34}^{(0)} = q_1^2 Z_{14} G_{1,0}\cos 2\theta,$$

$$A_{34}^{(1)} = \frac{q_1^2}{k_1}\bar{\mu}_{11}\left(H_{14} G_{1,0}\sin 2\theta + \frac{k_1}{\bar{\mu}_{11}} Z_{14} G_{1,1}\cos 2\theta\right),$$

$$A_{34}^{(2)} = \frac{q_1^2}{k_1}\bar{\mu}_{11}\left(H_{14} G_{1,1}\sin 2\theta + \frac{Z_{14} G_{1,0}\varepsilon_1}{k_1}\cos 2\theta\right),$$

$$A_{35}^{(0)} = -\frac{k_1}{\bar{\mu}_{11}} q_1^2 Z_{14}\sin 2\theta,$$

$$A_{35}^{(1)} = q_1^2 H_{14}\cos 2\theta, \quad A_{35}^{(2)} = 0, \qquad \text{(B15)}$$

$$A_{41}^{(0)} = \bar{\mu}_{11}\left(\frac{Z_{36}}{\bar{\mu}_{33}} q_1^2 - \frac{Z_{14}}{\bar{\mu}_{11}} k_1^2\right)\sin 2\theta,$$

$$A_{41}^{(1)} = \bar{\mu}_{11} k_1 H_{14}\cos 2\theta, \quad A_{41}^{(2)} = 0,$$

$$A_{42}^{(0)} = k_1 Z_{14}\sin 2\theta,$$

$$A_{42}^{(1)} = -\bar{\mu}_{11} H_{14}\cos 2\theta, \quad A_{42}^{(2)} = 0,$$

$$A_{43}^{(0)} = \bar{\mu}_{11}\Lambda, \quad A_{43}^{(1)} = 0, \quad A_{43}^{(2)} = 0,$$

$$A_{44}^{(0)} = -\bar{\mu}_{11}\left(\frac{Z_{36}}{\bar{\mu}_{33}} q_1^2 - \frac{Z_{14}}{\bar{\mu}_{11}} k_1^2\right)\cos 2\theta,$$

$$A_{44}^{(1)} = \bar{\mu}_{11} k_1 H_{14}\sin 2\theta, \quad A_{44}^{(2)} = 0,$$

$$A_{45}^{(0)} = 1, \quad A_{45}^{(1)} = 0, \quad A_{45}^{(2)} = -\bar{\mu}_{11}\varepsilon_1, \qquad \text{(B16)}$$

$$A_{51}^{(0)} = q_1^2 k_1^2 (H_{14} + H_{36})\sin 2\theta,$$

$$A_{51}^{(1)} = q_1^2 k_1 Z_{14}\varepsilon_1 \cos 2\theta, \quad A_{51}^{(2)} = -q_1^2 H_{36}\bar{\mu}_{11}\varepsilon_1 \sin 2\theta,$$

$$A_{52}^{(0)} = -q_1^2 k_1 H_{14}\sin 2\theta,$$

$$A_{52}^{(1)} = -q_1^2\varepsilon_1 Z_{14}\cos 2\theta; \quad A_{52}^{(2)} = 0,$$

$$A_{53}^{(0)} = \varepsilon_1 q_1^2 + \varepsilon_3 k_1^2, \quad A_{53}^{(1)} = 0, \quad A_{53}^{(2)} = -\bar{\mu}_{11}\varepsilon_1\varepsilon_3$$

$$A_{54}^{(0)} = -q_1^2 k_1^2 (H_{14} + H_{36})\cos 2\theta,$$

$$A_{54}^{(1)} = q_1^2 k_1 Z_{14}\varepsilon_1 \sin 2\theta, \quad A_{54}^{(2)} = q_1^2 H_{36}\bar{\mu}_{11}\varepsilon_1 \cos 2\theta,$$

$$A_{55}^{(0)} = -q_1^2 k_1^2 \Lambda, \quad A_{55}^{(1)} = 0, \quad A_{55}^{(2)} = q_1^2 \bar{\mu}_{11}\varepsilon_1 \Lambda. \quad \text{(B17)}$$

The determinant D_0 can be easily calculated if we write $A_{11}^{(0)}$, $A_{14}^{(0)}$, $A_{21}^{(0)}$, $A_{24}^{(0)}$, and $A_{32}^{(0)}$ as

$$A_{11}^{(0)} = -y^2 \sin 2\theta + a_{11}, \quad A_{14}^{(0)} = y^2 \cos 2\theta + a_{14},$$

$$A_{21}^{(0)} = -y^2 \cos 2\theta + a_{21}, \quad A_{24}^{(0)} = -y^2 \sin 2\theta + a_{24},$$

$$A_{32}^{(0)} = y^2 + b_{32}, \quad \text{(B18)}$$

and

$$a_{11} = \left(\tilde{f}_{44}k_1^2 + \frac{\tilde{c}_{11} + \tilde{c}_{12}}{2} q_1^2\right)\sin 2\theta,$$

$$a_{14} = -(\tilde{f}_{44}k_1^2 + \tilde{f}_{66}q_1^2)\cos 2\theta,$$

$$a_{21} = (\tilde{f}_{44}k_1^2 + \tilde{c}_{11}q_1^2)\cos 2\theta + k_1 Z_{14} H_{1,0},$$

$$a_{24} = \left(\tilde{f}_{44}k_1^2 + \frac{\tilde{c}_{11} - \tilde{c}_{12}}{2} q_1^2\right)\sin 2\theta + k_1 Z_{14} G_{1,0},$$

$$a_{32} = -\left(\tilde{f}_{44}q_1^2 + \tilde{c}_{33}k_1^2 + \frac{q_1^2}{k_1} Z_{14} H_{1,0}\cos 2\theta\right). \quad \text{(B19)}$$

For the remaining terms $A_{ij}^{(0)} = a_{ij}$. It is obtained

$$D_0 = y^6 + a_2 y^4 + a_1 y^2 + a_0, \quad \text{(B20)}$$

where

$$a_2 = -\sigma_1 + \frac{\tau_2 - \sigma_7 + \sigma_9}{\sigma_2},$$

$$a_1 = \tau_1 + \frac{\sigma_3 - \tau_2\sigma_1 - \sigma_4 + \sigma_5 - \sigma_6 + \sigma_8 - \tau_8}{\sigma_2},$$

$$a_0 = \frac{\tau_1\tau_2 - \tau_3 + \tau_4 - \tau_5 + \tau_6 - \tau_7 + \tau_9 + \tau_{10}}{\sigma_2}, \quad \text{(B21)}$$

and

$$\sigma_1 = (a_{11} + a_{24})\sin 2\theta + (a_{21} - a_{14})\cos 2\theta,$$

$$\sigma_2 = a_{43}a_{55} - a_{53},$$

$$\sigma_3 = a_{22}(a_{43}a_{55} - a_{53})(a_{34}\sin 2\theta + a_{31}\cos 2\theta),$$

$$\sigma_4 = a_{22}(a_{33}a_{55} - a_{53}a_{35})(a_{44}\sin 2\theta + a_{41}\cos 2\theta),$$

$$\sigma_5 = a_{22}(a_{33} - a_{43}a_{35})(a_{54}\sin 2\theta + a_{51}\cos 2\theta),$$

$$\tau_1 = a_{11}a_{24} - a_{21}a_{14},$$

$$\tau_2 = a_{32}\sigma_2 - a_{33}(a_{42}a_{55} - a_{52}) + a_{35}(a_{42}a_{53} - a_{52}a_{43}),$$

$$\tau_3 = a_{22}(a_{43}a_{55} - a_{53})(a_{34}a_{11} - a_{31}a_{14}),$$

$$\tau_4 = a_{22}(a_{33}a_{55} - a_{53}a_{35})(a_{44}a_{11} - a_{41}a_{14}),$$

$$\tau_5 = a_{22}(a_{33} - a_{43}a_{35})(a_{54}a_{11} - a_{51}a_{14}), \quad \text{(B22)}$$

$$p_1 = b_{44}\sin 2\theta - b_{41}\cos 2\theta,$$

$$r_1 = b_{44}b_{21} - b_{41}b_{24},$$

$$p_2 = b_{13}b_{55} - b_{53}b_{15},$$

$$r_2 = b_{12}(b_{33}b_{55} - b_{53}b_{35}) - b_{32}(b_{13}b_{55} - b_{53}b_{15}) + b_{52}(b_{13}b_{35} - b_{33}b_{15}),$$

$$\sigma_7 = p_1 p_2, \quad \sigma_8 = p_1 r_2 + p_2 r_1,$$

$$\tau_7 = r_1 r_2, \quad \text{(B23)}$$

$$S_1 = b_{12}(b_{43}b_{55} - b_{53}) - b_{13}(b_{42}b_{55} - b_{52}) + b_{15}(b_{42}b_{53} - b_{52}b_{43}),$$

$$\sigma_6 = S_1(b_{14}\cos 2\theta - b_{31}\sin 2\theta),$$

$$\tau_6 = S_1(b_{34}b_{21} - b_{31}b_{24}), \quad \text{(B24)}$$

$$p_3 = a_{54}\cos 2\theta - a_{51}\sin 2\theta,$$

$$r_3 = a_{54}a_{21} - a_{51}a_{24},$$

$$p_4 = a_{13} - a_{43}a_{15},$$

$$r_4 = a_{12}(a_{33} - a_{43}a_{35}) - a_{32}(a_{13} - a_{43}a_{15}) + a_{42}(a_{13}a_{35} - a_{33}a_{15}),$$

$$\sigma_9 = p_3 p_4, \quad \tau_8 = p_3 r_4 + p_4 r_3,$$

$$\tau_9 = r_3 r_4, \quad \text{(B25)}$$

$$\tau_{10} = -a_{22}(a_{31}a_{44} - a_{41}a_{34})(a_{13}a_{55} - a_{53}a_{15}) + a_{22}(a_{31}a_{54} - a_{51}a_{34})(a_{13} - a_{43}a_{15}) - a_{22}(a_{41}a_{54} - a_{51}a_{44})(a_{13}a_{35} - a_{33}a_{15}). \quad \text{(B26)}$$

APPENDIX C

In this appendix we discuss for the symmetry $4'22'$ the particular case $\partial_z f = \partial_y f = 0$ that can be treated analytically, without the approximate calculation of the acoustic and the electromagnetic frequencies. The matter displacements and the electromagnetic field depend only on x and t. The constitutive equations (3)–(5) and the Maxwell equations $\nabla\cdot\mathbf{D}=0$ and $\nabla\cdot\mathbf{B}=0$ give the set

$$\varepsilon_{11}\partial_x E_x+\lambda_{11}\partial_x H_x=0,$$

$$\mu_{11}\partial_x H_x+\lambda_{11}\partial_x E_x=0 \tag{C1}$$

whose solution is $\partial_x E_x=\partial_x H_x=0$ if $\varepsilon_{11}\mu_{11}\neq\lambda_{11}^2$. The fields E_x and H_x do not depend on the spatial coordinates. The mechanical equations of motion [the first three of Eqs. (7)] give then

$$c_{11}\partial_{xx}u=-\varrho\omega^2 u,$$

$$c_{66}\partial_{xx}v-e_{36}\partial_x E_z-z_{36}\partial_x H_z=-\varrho\omega^2 v,$$

$$c_{44}\partial_{xx}w-e_{14}\partial_x E_y-z_{14}\partial_x H_y=-\varrho\omega^2 w. \tag{C2}$$

The first equation says that the displacement u satisfies the acoustic wave equation with dispersion relation $\omega=(\frac{c_{11}}{\varrho})^{1/2}q_x$; furthermore, the displacement v is connected to the fields E_z and H_z (second equation) through the piezoelectric and the piezomagnetic coefficients e_{36} and z_{36} and the displacement w is connected to E_y and H_y through e_{14} and z_{14} (third equation). The other Maxwell equations give

$$-\partial_x E_z=i\omega B_y,\quad \partial_x E_y=i\omega B_z,$$

$$\partial_x H_z=i\omega D_y,\quad \partial_x H_y=-i\omega D_z. \tag{C3}$$

Using the constitutive equations to calculate H_z, H_y, D_z, and D_y and substituting in the Fourier transform of Eqs. (C2) and (C3), the following are obtained:

$$(x^2-\tilde{f}_{66}q_x^2)\tilde{v}+i\beta H_{36}\frac{\tilde{B}_y}{\sqrt{c_{44}\mu_0}}-iq_x\frac{Z_{36}}{\bar{\mu}_{33}}\frac{\tilde{B}_z}{\sqrt{c_{44}\mu_0}}=0,$$

$$(x^2-\tilde{f}_{44}q_x^2)\tilde{w}-i\beta H_{14}\frac{\tilde{B}_z}{\sqrt{c_{44}\mu_0}}-iq_x\frac{Z_{14}}{\bar{\mu}_{11}}\frac{\tilde{B}_y}{\sqrt{c_{44}\mu_0}}=0,$$

$$-\beta q_x H_{36}\tilde{v}+\frac{Z_{14}}{\bar{\mu}_{11}}q_x^2\tilde{w}+i\frac{q_x^2-\varepsilon_3\bar{\mu}_{11}\beta^2}{\bar{\mu}_{11}q_x}\frac{\tilde{B}_y}{\sqrt{c_{44}\mu_0}}+i\beta\Lambda\frac{\tilde{B}_z}{\sqrt{c_{44}\mu_0}}=0,$$

$$\frac{Z_{36}}{\bar{\mu}_{33}}q_x^2\tilde{v}+\beta q_x H_{14}\tilde{w}+i\beta\Lambda\frac{\tilde{B}_y}{\sqrt{c_{44}\mu_0}}+i\frac{q_x^2-\varepsilon_1\bar{\mu}_{33}\beta^2}{\bar{\mu}_{33}q_x}\frac{\tilde{B}_z}{\sqrt{c_{44}\mu_0}}=0. \tag{C4}$$

The eigenfrequencies are now obtained through the equation

$$\begin{aligned}
&a_1a_3\left(-\frac{\Delta_1\Delta_3}{q_x^2\bar{\mu}_{11}\bar{\mu}_{33}}+\beta^2\Lambda^2\right)-a_1\frac{Z_{14}}{\bar{\mu}_{11}}q_x^2\left(\frac{\Delta_1 Z_{14}}{\bar{\mu}_{11}\bar{\mu}_{33}}-\beta^2H_{14}\Lambda\right)\\
&+a_1\beta^2H_{14}\left(H_{14}\Lambda q_x^2-\frac{H_{14}\Delta_3}{\bar{\mu}_{11}}\right)\\
&-a_3\beta^2H_{36}\left(\frac{H_{36}\Delta_1}{\bar{\mu}_{33}}+\frac{Z_{36}\Lambda}{\bar{\mu}_{33}}q_x^2\right)\\
&-a_3q_x^2\frac{Z_{36}}{\bar{\mu}_{33}}\left(\beta^2H_{36}\Lambda+\frac{Z_{36}\Delta_3}{\bar{\mu}_{33}\bar{\mu}_{11}}\right)\\
&-q_x^2\left(\beta^2H_{36}H_{14}+\frac{Z_{36}Z_{14}}{\bar{\mu}_{33}\bar{\mu}_{11}}q_x^2\right)^2=0,
\end{aligned} \tag{C5}$$

where

$$a_1=x^2-\tilde{f}_{66}q_x^2,\quad a_3=x^2-\tilde{f}_{44}q_x^2,$$

$$\Delta_1=q_x^2-\varepsilon_1\bar{\mu}_{33}\beta^2,\quad \Delta_3=q_x^2-\varepsilon_3\bar{\mu}_{11}\beta^2. \tag{C6}$$

The fourth-order Eq. (C5) cannot be reduced in a simpler form even if the piezoelectric, the piezomagnetic, and the magnetoelectric coefficients are zero. In this particular case, the eigenfrequencies are $a_3=0$ [i.e., $\omega=(\frac{c_{44}}{\varrho})^{1/2}q_x$], $a_1=0$ [i.e., $\omega=(\frac{c_{66}}{\varrho})^{1/2}q_x$], $\Delta_3=0$ (i.e., $\omega=\frac{c}{\sqrt{\varepsilon_3\bar{\mu}_{11}}}q_x$), and $\Delta_1=0$ (i.e., $\omega=\frac{c}{\sqrt{\varepsilon_1\bar{\mu}_{33}}}q_x$), i.e., two modes are acoustic and two are electromagnetic. The dielectric constants are positive only if $\varepsilon_{11}\mu_{11}>\lambda_{11}^2$ and $\varepsilon_{33}\mu_{33}>\lambda_{33}^2$. Furthermore, Eq. (C5) contains renormalized elastic and piezoelectric coefficients and also terms in β^2, i.e., of the order of α^2. This might be the indication that the acoustic frequencies are modified even to the zeroth order in α. But it is easily shown that this does not occur because to the zeroth-order Eq. (C5) becomes

$$\begin{aligned}
&a_1a_3+a_1\frac{Z_{14}^2}{\bar{\mu}_{11}}q_x^2+a_3\frac{Z_{36}^2}{\bar{\mu}_{33}}q_x^2+\frac{Z_{14}^2}{\bar{\mu}_{11}}\frac{Z_{36}^2}{\bar{\mu}_{33}}q_x^4\\
&=\left(a_1+\frac{Z_{36}^2}{\bar{\mu}_{33}}q_x^2\right)\left(a_3+\frac{Z_{14}^2}{\bar{\mu}_{11}}q_x^2\right)=0,
\end{aligned} \tag{C7}$$

so that all the renormalizing terms disappear. On the other hand, if we divide Eq. (C5) by a_1a_3 it is found that the zeroth-order electromagnetic solutions are given by $\Delta_1=0$ and $\Delta_3=0$ and the first corrective term is of the order α^2.

In conclusion, one mode is purely acoustic and concerns the matter displacement along the x axis; the other two piezoacoustic (electromagnetic) modes are coupled to the electromagnetic (piezoacoustic) field only through terms of the order α^2. Analogous results are found in the cases $\partial_x f=\partial_y f=0$ and $\partial_x f=\partial_z f=0$.

APPENDIX D

1. Case sin 2θ=0

In this case the wave vector has components along the x and the z axes or along the y and the z axes. The matrix elements (31)–(35) assume a much simpler form, yet the problem cannot be analytically solved. Further constraints must be imposed. If, for instance, $z_{ij}=\lambda_{ij}=0$ (purely piezoelectric material), then $f_{44}=c_{44}$, $f_{66}=c_{66}$, $h_{14}=e_{14}$, $h_{36}=e_{36}$ and the coefficients A_{15}, A_{23}, A_{43}, and A_{55} are zero. The wave vector has components only along the x and the z axes or along the y and the z axes and the eigenfrequencies are found through the equation

$$[A_{13}A_{54}-A_{53}A_{41}][A_{21}(A_{32}A_{45}+A_{42}A_{35}) + A_{22}(A_{31}A_{45}-A_{41}A_{35}) - A_{25}(A_{31}A_{42}+A_{41}A_{32})]=0. \tag{D1}$$

When the factor in the first row of Eq. (D1) is zero, the quantities $\tilde{M}$, $\tilde{\psi}$, and $\tilde{G}$ are not zero, but $\tilde{\varphi}=\tilde{W}=\tilde{C}_1=0$. This implies that $\tilde{B}_z=0$, $\tilde{E}_z\neq 0$, $F_x=\partial_x H$ (i.e., $\tilde{F}_x=iq_x\tilde{H}$), $F_y=\partial_y H$ (i.e., $\tilde{F}_y=iq_y\tilde{H}$), $u=\partial_y\psi$ (i.e., $\tilde{u}=iq_y\tilde{\psi}$), $v=-\partial_x\psi$ (i.e., $\tilde{v}=-iq_x\tilde{\psi}$), and $\partial_x u+\partial_y v=0$. The acoustic and the electromagnetic eigenfrequencies are

$$x^2 \simeq k^2+\bar{c}_{66}q^2+q^2k^2\frac{(E_{14}+E_{36})^2}{\bar{\varepsilon}_{11}q^2+\bar{\varepsilon}_{33}k^2},$$

$$\beta^2 \simeq \frac{1}{\bar{\mu}_{11}}\left(\frac{q^2}{\bar{\varepsilon}_{33}}+\frac{k^2}{\bar{\varepsilon}_{11}}\right), \tag{D2}$$

showing that the piezoelectric coefficients modify the acoustic frequency even to the zeroth order in α.

When the second factor of Eq. (D1) is zero, $\tilde{M}=\tilde{\psi}=\tilde{G}=0$ and $\tilde{\varphi}$, $\tilde{W}$, and $\tilde{C}_1$ are not zero. This implies that $\tilde{B}_z\neq 0$, $\tilde{E}_z=\tilde{F}_z=\tilde{\phi}=0$, $F_x=\partial_y C_1$ (i.e., $\tilde{F}_x=iq_y\tilde{C}_1$), $F_y=-\partial_x C_1$ (i.e., $\tilde{F}_y=-iq_x\tilde{C}_1$), $u=\partial_x\varphi$ (i.e., $\tilde{u}=iq_x\tilde{\varphi}$), $v=\partial_y\varphi$ (i.e., $\tilde{v}=iq_y\tilde{\varphi}$), and $\partial_y u-\partial_x v=0$, $\nabla\cdot\mathbf{E}=0$. The frequencies are found by solving the third-order algebraic equation discussed in Sec. IV.

Another example is the case in which $e_{ij}=\lambda_{ij}=0$ (purely piezomagnetic material). This implies that $h_{14}=0$, $h_{36}=0$ and the coefficients A_{13}, A_{25}, A_{34}, A_{43}, and A_{55} are zero. The eigenfrequencies are given by

$$[A_{14}A_{45}-A_{44}A_{15}][A_{21}(A_{32}A_{53}-A_{33}A_{52}) + A_{22}(A_{31}A_{53}-A_{33}A_{51}) - A_{23}(A_{31}A_{52}-A_{51}A_{32})]=0. \tag{D3}$$

When the factor in the first row of Eq. (D3) is zero, $\tilde{\psi}$ and $\tilde{C}_1$ are not zero, but $\tilde{\varphi}=\tilde{W}=\tilde{M}=\tilde{G}=0$. This implies that $\tilde{B}_z\neq 0$, $\tilde{E}_z=0$, $\tilde{F}_z=\tilde{\phi}=0$, $F_x=\partial_y C_1$ (i.e., $\tilde{F}_x=iq_y\tilde{C}_1$), $F_y=-\partial_x C_1$ (i.e., $\tilde{F}_y=-iq_x\tilde{C}_1$), $u=\partial_y\psi$ (i.e., $\tilde{u}=iq_y\tilde{\psi}$), $v=-\partial_x\psi$ (i.e., $\tilde{v}=-iq_x\tilde{\psi}$), and $\partial_x u+\partial_y v=0$, $\nabla\cdot\mathbf{E}=0$. The acoustic and the electromagnetic frequencies are

$$x^2 \simeq \tilde{f}_{44}k^2+\tilde{f}_{66}q^2-\frac{1}{\bar{\mu}_{11}(k^2+q^2)}\left(Z_{14}k^2-\frac{\bar{\mu}_{11}Z_{36}}{\bar{\mu}_{33}}q^2\right)^2,$$

$$\beta^2 \simeq \frac{1}{\bar{\mu}_{11}\bar{\varepsilon}_{11}}(q^2+k^2), \tag{D4}$$

showing again that the acoustic frequency is modified to the zeroth order in α by the piezomagnetic coefficients, but it appears also the possibility that such mode becomes instable when $x^2\leq 0$.

When the second factor of Eq. (D3) is zero, we have $\tilde{\psi}=\tilde{C}_1=0$, $\tilde{\varphi}$, $\tilde{W}$, $\tilde{M}$ are not zero and $B_z=0$, $\tilde{E}_z\neq 0$. This implies that $u=\partial_x\varphi$ (i.e., $\tilde{u}=iq_x\tilde{\varphi}$), $v=\partial_y\varphi$ (i.e., $\tilde{v}=-iq_y\tilde{\varphi}$), and $\partial_y u-\partial_x v=0$. $F_x=\partial_x H$ (i.e., $\tilde{F}_x=iq_x\tilde{H}$), $F_y=\partial_x H$ (i.e., $\tilde{F}_y=iq_y\tilde{H}$). The third-order algebraic equation is discussed in Sec. IV.

2. Case cos 2θ=0

In this case the wave vector has components $|q_x|=|q_y|$ and k not zero. For purely piezoelectric materials the eigenfrequencies are found through the equation

$$[A_{24}A_{45}-A_{44}A_{25}][A_{11}(A_{32}A_{53}-A_{52}A_{33}) + A_{12}(A_{31}A_{53}-A_{33}A_{51}) - A_{13}(A_{31}A_{52}-A_{51}A_{32})]=0. \tag{D5}$$

When the factor in the first row of Eq. (D5) is zero, we find that $\tilde{\psi}$ and $\tilde{C}_1$ are not zero, but $\tilde{\varphi}=\tilde{W}=\tilde{M}=\tilde{G}=0$. This implies that $\tilde{B}_z\neq 0$, $\tilde{E}_z=0$, $\tilde{F}_z=\tilde{\phi}=0$, $F_x=\partial_y C_1$ (i.e., $\tilde{F}_x=iq_y\tilde{C}_1$), $F_y=-\partial_x C_1$, (i.e., $\tilde{F}_y=-iq_x\tilde{C}_1$), $u=\partial_y\psi$ (i.e., $\tilde{u}=iq_y\tilde{\psi}$), $v=-\partial_x\psi$ (i.e., $\tilde{v}=-iq_x\tilde{\psi}$), and $\partial_x u+\partial_y v=0$, $\nabla\cdot\mathbf{E}=0$. The acoustic and the electromagnetic frequencies are to the zeroth order in α,

$$x^2 \simeq k^2+\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}q^2,$$

$$\beta^2 \simeq \frac{k^2+q^2}{\bar{\mu}_{11}\bar{\varepsilon}_{11}}. \tag{D6}$$

The piezoelectric corrections are of the second order in α.

When the second factor of Eq. (D5) is zero, we have $\tilde{\psi}=\tilde{C}_1=0$, $\tilde{\varphi}$, $\tilde{W}$, $\tilde{M}$ are not zero, and $B_z=0$, $\tilde{E}_z\neq 0$. This implies that $u=\partial_x\varphi$ (i.e., $\tilde{u}=iq_x\tilde{\varphi}$), $v=\partial_y\varphi$ (i.e., $\tilde{v}=-iq_y\tilde{\varphi}$), and $\partial_y u-\partial_x v=0$. $F_x=\partial_x H$ (i.e., $\tilde{F}_x=iq_x\tilde{H}$), $F_y=\partial_x H$ (i.e., $\tilde{F}_y=iq_y\tilde{H}$). The third-order equation gives, in this case, the dependence of the frequencies on the piezoelectric parameters, as it is shown in Sec. IV.

For purely piezomagnetic materials the eigenfrequencies are given by

$$[A_{23}A_{54}-A_{24}A_{53}][A_{11}A_{32}A_{45}-A_{12}(A_{31}A_{45}-A_{41}A_{35}) \times[-A_{15}A_{32}A_{41}]=0. \tag{D7}$$

When the first factor of Eq. (D7) is zero, the quantities $\tilde{M}$, $\tilde{\psi}$, and $\tilde{G}$ are not zero, but $\tilde{\varphi}=\tilde{W}=\tilde{C}_1=0$. This implies that $\tilde{B}_z=0$, $\tilde{E}_z\neq 0$, $F_x=\partial_x H$ (i.e., $\tilde{F}_x=iq_x\tilde{H}$), $F_y=\partial_y H$ (i.e., $\tilde{F}_y=iq_y\tilde{H}$), $u=\partial_y\psi$ (i.e., $\tilde{u}=iq_y\tilde{\psi}$), $v=-\partial_x\psi$ (i.e., $\tilde{v}=-iq_x\tilde{\psi}$), and $\partial_x u+\partial_y v=0$. The acoustic and the electromagnetic eigenfrequencies are to the zeroth order in α,

$$x^2 \simeq k^2+\frac{\tilde{c}_{11}-\tilde{c}_{12}}{2}q^2,$$

$$\beta^2 \simeq \frac{\bar{\varepsilon}_{11}q^2 + \bar{\varepsilon}_{33}k^2}{\bar{\mu}_{11}\bar{\varepsilon}_{11}\bar{\varepsilon}_{33}}, \tag{D8}$$

showing that the piezomagnetic correction to the acoustic and electromagnetic modes is of the order α^2.

When the second factor of Eq. (D7) is zero, $\tilde{M}=\tilde{\psi}=\tilde{G}=0$ and $\tilde{\varphi}$, $\tilde{W}$, and $\tilde{C}_1$ are not zero. This implies that $\tilde{B}_z\neq 0$, $\tilde{E}_z=\tilde{F}_z=\tilde{\phi}=0$, $F_x=\partial_y C_1$ (i.e., $\tilde{F}_x=iq_y\tilde{C}_1$), $F_y=-\partial_x C_1$ (i.e., $\tilde{F}_y=-iq_x\tilde{C}_1$), $u=\partial_x\varphi$ (i.e., $\tilde{u}=iq_x\tilde{\varphi}$), $v=\partial_y\varphi$ (i.e., $\tilde{v}=iq_y\tilde{\varphi}$), and $\partial_y u-\partial_x v=0$, $\nabla\cdot\mathbf{E}=0$. Also, this case is discussed in Sec. IV.

[1] W. P. Mason, *Piezoelectric Crystals and Their Applications to Ultrasonics* (D. Van Nostrand, Co., New York, 1949).
[2] W. G. Cady, *Piezoelectricity* (Dover Publications, Inc., New York, 1962).
[3] H. F. Tiersten, *Linear Piezoelectric Plate Vibrations* (Plenum Press, New York, 1969).
[4] B. A. Auld, *Acoustic Fields and Waves in Solids* (John Wiley & Sons, New York, 1973), Vol. I; *Acoustic Fields and Waves in Solids* (John Wiley & Sons, New York, 1973), Vol. II.
[5] J. Yang, *An Introduction to the Theory of Piezoelectricity* (Springer, New York, 2005).
[6] A. S. Borovik-Romanov, Sov. Phys. JETP **11**, 786 (1960).
[7] See, for instance, C. J. Bradley and A. P. Cracknell, *The Mathematical Theory of Symmetry in Solids* (Clarendon Press, Oxford, 1972).
[8] I. E. Dzyaloshinskii, Sov. Phys. JETP **6**, 621 (1958).
[9] I. F. Mirsaev, Phys. Solid State **43**, 1527 (2001).
[10] L. G. Fel, J. Math. Phys. **43**, 2606 (2002).
[11] S. J. Hyder, M. Sunar, and F. Mahamood, Proc. Inst. Mech. Eng., Part I: J. Syst. Control Eng. **218**, 27 (2003).
[12] H. Katsura, N. Nagaosa, and A. V. Balatsky, Phys. Rev. Lett. **95**, 057205 (2005).
[13] G. Nenert and T. T. M. Palstra, J. Phys.: Condens. Matter **19**, 406213 (2007).
[14] N. Zhang, X. Fang, and Z. L. Wang, J. Appl. Phys. **104**, 014504 (2008).
[15] Simon A. J. Kimber and J. Paul Attfield, J. Mater. Chem. **17**, 4885 (2007).
[16] D. Khomskii, Phys. **2**, 20 (2009).
[17] J. J. Kyame, J. Acoust. Soc. Am. **21**, 159 (1949).
[18] A. R. Hutson and D. L. White, J. Appl. Phys. **33**, 40 (1962).
[19] R. D. Mindlin, Int. J. Solids Struct. **9**, 697 (1973).
[20] P. C. Y. Lee, Y.-G. Kim, and J. H. Prevost, J. Appl. Phys. **67**, 6633 (1990); P. C. Y. Lee, *ibid.* **65**, 1395 (1989).
[21] J. S. Yang, Int. J. Appl. Electromagn. Mech. **12**, 235 (2000).
[22] J. S. Yang, Acta Mech. **168**, 111 (2004).
[23] F. C. Moon, J. Acoust. Soc. Am. **48**, 253 (1970).
[24] G. J. Bush and G. P. Carman, Int. J. Mech. Mater. Des. **4**, 105 (2008).
[25] H. Liu, S. N. Zhu, G. Dong, Y. Y. Zhu, Y. F. Cheng, and N. B. Ming, Phys. Rev. B **71**, 125106 (2005).
[26] G. Iadonisi, C. A. Perroni, G. Cantele, and D. Ninno, J. Appl. Phys. **103**, 064109 (2008).
[27] H. S. Paul, J. Acoust. Soc. Am. **40**, 1077 (1966).
[28] J. Yu, Q. Ma, and S. Su, Ultrasonics **48**, 664 (2008).